$E(n)$-local Greek letter elements $\alpha^{(n)}_{t/a}$ for $2p-1=n^2$

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Let $E(n)$ be the nth Johnson-Wilson spectrum at a prime number p. In this note, we investigate Greerk letter elements of the type $\alpha^{(n)}_{t/a}$ in the homotopy groups $\pi_*(L_nS^0)$ of the $E(n)$-localized sphere spectrum for $2p-1=n^2$. In particular, at $(p,n)=(5,3)$, the main theorem implies that if $125\nmid t$, then any $\gamma_{t/a}\left(=\alpha^{(3)}_{t/a}\right)$ exists in $\pi_*(L_3S^0)$, and if $125\mid t$, then $\gamma_{t/a}$ for $a\le 26$ exists in $\pi_*(L_3S^0)$.

1. Introduction

Let p be a prime number and $\mathcal{S}_p$ the stable homotopy category of p-local spectra. The nth Johnson-Wilson spectrum $E(n)$ represents the homology theory $E(n)_*(-)$, whose coefficient algebra is

$$E(n)_*=\mathbb{Z}_{(p)}[v_1,\dots,v_{n-1},v_n^{\pm1}]\quad\text{with}\quad |v_i|=2(p^i-1).$$

For the Bousfield localization functor $L_n\colon\mathcal{S}_p\to\mathcal{S}_p$ with respect to $E(n)$, we denote $\mathcal{L}_n=L_n(\mathcal{S}_p)$. Since $L_{n-1}(\mathcal{L}_n)=\mathcal{L}_{n-1}$, we have the universal homomorphism

$$u_X\colon\pi_*(X)\to\lim_n\pi_*(L_nX)$$

for a spectrum X. By the chromatic convergence theorem of Hopkins-Ravenel (*cf.* [3, Th. 7.5.7]), if X is a finite spectrum, then the homomorphism u_X is an isomorphism. In particular, $\pi_*(S^0)=\lim_n\pi_*(L_nS^0)$ where S^0 is the sphere spectrum. For a spectrum X, we have the $E(n)$-based Adams spectral sequence

$$E_2^{s,t}=\mathrm{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*,E(n)_*(X))\Rightarrow\pi_{t-s}(L_nX).$$

Hereafter, we denote by $E(n)_r^{s,t}(X)$ the E_r-term of the spectral sequence. For an $E(n)_*(E(n))$-comodule M, we abbreviate

$$H^{s,t}M=\mathrm{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*,M)\quad\text{and}\quad H^sM=\bigoplus_t H^{s,t}M.$$

For $k<n$, we let I_k be the ideal $(p,v_1,\dots,v_{k-1})$ of $E(n)_*$, and

$$N_k=E(n)_*/I_k.$$

We also put $I_n^{(a)}=(I_{n-1},v_{n-1}^a)$. The short exact sequences

$$0\to N_{n-1}\xrightarrow{v_{n-1}^a}N_{n-1}\to E(n)_*/I_n^{(a)}\to 0$$

and

$$0\to N_k\xrightarrow{v_k}N_k\to N_{k+1}\to 0$$

give rise to the connecting homomorphisms

$$\delta_a\colon H^sE(n)_*/I_n^{(a)}\to H^{s+1}N_{n-1}\quad\text{and}\quad\partial_k\colon H^sN_{k+1}\to H^{s+1}N_k,$$

respectively. If v_n^t is in $H^0E(n)_*/I_n^{(a)}$, then we define

$$\overline{\alpha}^{(n)}_{t/a}=\partial_0\cdots\partial_{n-2}\delta_a(v_n^t)\in H^nN_0=E(n)_2^{n,*}(S^0).$$

Traditionally, for a small n, we use the nth Greek letter instead of $\alpha^{(n)}$, that is,

$$\overline{\alpha}_{t/a}=\overline{\alpha}^{(1)}_{t/a},\quad\overline{\beta}_{t/a}=\overline{\alpha}^{(2)}_{t/a},\quad\overline{\gamma}_{t/a}=\overline{\alpha}^{(3)}_{t/a},\quad\overline{\delta}_{t/a}=\overline{\alpha}^{(4)}_{t/a},\quad\overline{\varepsilon}_{t/a}=\overline{\alpha}^{(5)}_{t/a},\quad\dots.$$

If $\overline{\alpha}^{(n)}_{t/a}$ converges to an element $x\in\pi_*(L_nS^0)$, we denote it by $\alpha^{(n)}_{t/a}$. For $p>2$ and $n>2$, by [2, Th. 5.10], the element $\overline{\alpha}^{(n)}_{t/a}$ $(t\ne0)$ belongs to $E(n)_2^{n,*}(S^0)$ if and only if

(1)
$$a\le a(t)=\begin{cases}1 & \nu_p(t)=0,\\ pa(t/p) & \nu_p(t)=1\text{ or }0<\nu_p(t)\not\equiv1\mod(n-1),\\ pa(t/p)+p-1 & 1<\nu_p(t)\equiv1\mod(n-1).\end{cases}$$

Here, $\nu_p(t)=\max\{i\in\mathbb{Z}\colon p^i\mid t\}$. We immediately obtain the following theorem from the work of Shimomura-Yokotani [5]:

Theorem 2 (Corollary of [5, Th. 1.2]) If $2p\ge n^2+n$, then any $\overline{\alpha}^{(n)}_{t/a}\in E(n)_2^{n,*}(S^0)$ converges to $\alpha^{(n)}_{t/a}$ of $\pi_*(L_nS^0)$.

We recall that the Smith-Toda spectrum $V(k)$ satisfies

$$E(n)_*(V(k))=E(n)_*/I_{k+1}.$$

In this note, we show the following:

Theorem 3 Assume that

- $2p-1=n^2$,
- $V(n-2)$ exists, and

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- $V(n-2)$ admits a v_{n-1}-self map $v_{n-1}^e\colon \Sigma^{e|v_{n-1}|}V(n-2) \to V(n-2)$.

Then, the element $\alpha_{t/a}^{(n)}$ $(t \neq 0)$ exists in $\pi_*(L_nS^0)$ if

$$\begin{aligned} a &= ke \\ &\leq \min\left\{a(t),\ 1+\frac{p^2(p^{n-2}-1)}{p-1}\right\} \\ &= \begin{cases} p^{v_p(t)} & v_p(t) < n \\ 1+\frac{p^2(p^{n-2}-1)}{p-1} & v_p(t) \geq n \end{cases} \end{aligned}$$

for $k \geq 1$.

For example, the following cases satisfy $2p-1 = n^2$:

$$(p,n) = (5,3),\ (13,5),\ (41,9),\ \ldots.$$

We recall that, at $p = 5$, the spectrum $V(1)$ exists and it admits $v_2\colon \Sigma^{48}V(1) \to V(1)$ [6, Th. 1.4]. Thus, Theorem 3 at $(p,n) = (5,3)$ is

Corollary 4　At $p = 5$, the element $\gamma_{t/a}$ exists in $\pi_*(L_3S^0)$ if

$$a \leq \min\{a(t), 26\} = \begin{cases} 5^{v_5(t)} & v_5(t) < 3, \\ 26 & v_5(t) \geq 3. \end{cases}$$

Furthermore, in the last section, we consider the case for $(p,n) = (13,5)$.

2.　Proof of Theorem 3

We assume that the Smith-Toda spectrum $V(n-2)$ exists and the spectrum admits a v_{n-1}-self map

$$v_{n-1}^e\colon \Sigma^{e|v_{n-1}|}V(n-2) \to V(n-2).$$

By the periodicity theorem of Hopkins-Smith (*cf.* [3, Chapter 6]), the self map v_{n-1}^e exists if e is a sufficiently large. Let

$$q = 2p-2,$$

and we put $I_{n-1} = (p, v_1, \ldots, v_{n-2}) \subset E(n)_*$ and $I_n^{(a)} = (I_{n-1}, v_{n-1}^a) \subset E(n)_*$. We also denote $K(n)_* = E(n)_*/I_n^{(1)} = \mathbb{Z}/p[v_n^{\pm 1}]$. A spectrum V_{ke} is defined by the cofiber sequence

$$\Sigma^{ke|v_{n-1}|}V(n-2) \xrightarrow{(v_{n-1}^e)^k} V(n-2) \to V_{ke} \to \Sigma^{ke|v_{n-1}|+1}V(n-2),$$

and $E(n)_*(V_{ke}) = E(n)_*/I_n^{(ke)}$.

Lemma 5　Assume that $q+1 = n^2$. If $H^{q+1,t}E(n)_*/I_n^{(a)} \neq 0$, then $t \equiv c|v_2| \mod (|v_3|)$ with $0 \leq c < a$.

Proof. We recall that $H^0E(n)_*/I_n^{(1)} = H^0K(n)_* = K(n)_*$. Since the condition $q+1 = n^2$ implies $(p-1) \nmid n$, the Poincare duality implies $H^{q+1}E(n)_*/I_n^{(1)} = K(n)_*\{g\}$ with $|g| = 0$. Hence, if $H^{q+1,t}E(n)_*/I_n^{(1)} \neq 0$, then $t \equiv 0 \mod (|v_n|)$, and we see the claim at $a = 1$.

We prove the lemma by an induction for a. Assume that

If $H^{q+1,t}E(n)_*/I_n^{(a-1)} \neq 0$, then $t \equiv c|v_{n-1}| \mod (|v_n|)$ with $0 \leq c < a-1$.

We consider the exact sequence

$$\begin{aligned} H^{q+1,t-|v_{n-1}|}E(n)_*/I_n^{(a-1)} &\xrightarrow{v_{n-1}} H^{q+1,t}E(n)_*/I_n^{(a)} \\ &\to H^{q+1,t}E(n)_*/I_n^{(1)}. \end{aligned}$$

If $H^{q+1,t}E(n)_*/I_n^{(a)} \neq 0$, then $H^{q+1,t-|v_{n-1}|}E(n)_*/I_n^{(a-1)} \neq 0$ or $H^{q+1,t}E(n)_*/I_n^{(1)} \neq 0$. Therefore, by the assumption, $t-|v_{n-1}| \equiv c|v_{n-1}| \mod (|v_n|)$ with $0 \leq c < a-1$, or $t \equiv 0 \mod (|v_3|)$. Hence we have $t \equiv c|v_{n-1}| \mod (|v_n|)$ with $0 \leq c < a$. □

Proof of Theorem 3. If $ke \leq a(t)$ where $a(t)$ is in (1), then v_n^t is in $E(n)_2^{0,t|v_n|}(V_{ke})$. By the Morava vanishing theorem, the assumption $q+1 = 2p-1 = n^2$ implies that we have only possibly nonzero differential

$$d_{q+1}\colon E(n)_2^{0,t|v_n|}(V_{ke}) \to E(n)_2^{q+1,q+t|v_n|}(V_{ke}).$$

Put

$$e(i) = \frac{p^i-1}{p-1}$$

and then $|v_i| = e(i)q$. By Lemma 5, if $E(n)_2^{q+1,q+t|v_n|}(V_{ke}) \neq 0$, then $q+t|v_n| \equiv c|v_{n-1}| \mod (|v_n|)$ with $0 \leq c < ke$. This implies $1 \equiv ce(n-1) \mod (e(n))$. Since $pe(n-1) = e(n)-1$, we have

$$c \equiv -c(e(n)-1) = -cpe(n-1) \equiv -p \equiv e(n)-p \mod (e(n)).$$

However, the assumption $a = ke \leq 1+p^2(p^{n-2}-1)/(p-1) = e(n)-p$ implies that we have no c such that $0 \leq c < ke$ and $c \equiv e(n)-p \mod (e(n))$. Therefore, $E(n)_2^{q+1,q+t|v_n|}(V_{ke}) = 0$, and so v_n^t survives to $v \in \pi_*(L_nV_{ke})$. Let J denote the collapsing map from V_{ke} to the top cell. By the geometric boundary theorem, the composite $J \circ v$ is $\alpha_{t/a}^{(n)}$. □

Remark 6　In the case for $a = 2+p^2(p^{n-2}-1)/(p-1) = e(n)-p+1$, we cannot prove the existence of $\alpha_{t/e(n)-p+1}^{(n)}$ in the way of this note. Indeed, if $V_{e(n)-p+1}$ exists and $v_n^t \in E(n)_2^{0,t|v_n|}(V_{e(n)-p+1})$, then

$$d_{q+1}(v_n^t) \in E(n)_2^{q+1,t|v_n|+q}(V_{e(n)-p+1}) \ni v_{n-1}^{e(n)-p}v_n^{t-e(n-1)+1}g.$$

Therefore, we don't know whether or not $d_{q+1}(v_n^t) = 0$.

3.　Remarks in the case for $(p,n) = (13,5)$

By [6, Th. 1.1], at $(p,n) = (13,5)$, the spectrum $V(n-2) = V(3)$ exists. By the periodicity theorem of Hopkins-Smith (*cf*, [3, Chapter 6]), $V(3)$ admits a v_4-self map

$$v_4^{e_4}\colon \Sigma^{57120e_4}V(3) \to V(3)$$

for a sufficiently large e_4. Therefore, Theorem 3 implies that

Corollary 7　At $p = 13$, the element $\varepsilon_{t/a}$ exists in $\pi_*(L_5S^0)$ if

$$a = ke_4 \leq \min\{a(t), 30928\} = \begin{cases} 13^{\nu_{13}(t)} & \nu_{13}(t) < 5 \\ 30928 & \nu_{13}(t) \geq 5 \end{cases}$$

for $k \geq 1$.

Conjecture 8　$e_4 = 1$, that is, the spectrum $V(3)$ admits

$$v_4 \colon \Sigma^{57120} V(3) \to V(3)$$

at $p = 13$.

Remark that, by [1, Th. 1.3], we know that $V(7)$ doesn't exist at $p = 13$. Even if Conjecture 8 is true, we have no contradiction to this fact.

Corollary 9　If Conjecture 8 holds, then, at $p = 13$, the element $\varepsilon_{t/a}$ exists in $\pi_*(L_5S^0)$ if

$$a \leq \min\{a(t), 30928\} = \begin{cases} 13^{\nu_{13}(t)} & \nu_{13}(t) < 5, \\ 30928 & \nu_{13}(t) \geq 5. \end{cases}$$

References

[1] L. S. Nave, The Smith-Toda complex $V((p+1)/2)$ does not exist, Ann. of Math. **171** (2010), 491–509.

[2] H. R. Miller, D. C. Ravenel, W. S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, Ann. of Math. **106** (1977), 469–516.

[3] D. C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, Ann. of Math. Studies **128**, Princeton Univ. Press 1992.

[4] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres, Second edition*, AMS Chelsea Publishing, Providence RI. 2004.

[5] K. Shimomura, M. Yokotani, Existence of the Greek letter elements in the stable homotopy groups of $E(n)_*$-localized spheres, Publ. Res. Inst. Math. Sci. **30** (1994), 139–150.

[6] H. Toda, On spectra realizing exterior parts of the Steenrod algebra, Topology **10** (1971), 53–65.