The Remainder of Samuel-Smirnov Compactification of the Half-Line

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In this article, it is considered homomorphisms on unital vector lattices consisting of uniformly continuous real-valued functions. After summarizing lattice theoretic properties of homomorphisms, we explain how to construct compactifications as spaces of homomorphisms. We give a detailed proof of the characterization theorem of the Samuel-Smirnov compactification of the half-line given by F. and J. Cabello Sánchez [3]. Then we apply it to prove a structure theorem for the remainder, which provides a concrete description of the half-line version of the structure theorem given by R. Grant Woods [7].

Throughout this article, $\mathbb{H}$ denotes the half-line $[0,\infty)$ with the metric given by the absolute value $|x-y|$, $x,y\in\mathbb{H}$, and $\mathbb{N}$ denotes the space of natural numbers with the induced metric. Also, $X=(X,d)$ is assumed to be a metric space.

Let $\mathcal{L}\subset C(X)$ be a *unital* vector lattice, that is, $\mathcal{L}$ contains the unit $1 : X\to\mathbb{R}$. The sublattice of all bounded functions of $\mathcal{L}$ is denoted by $\mathcal{L}^*$. A function $\phi:\mathcal{L}\to\mathbb{R}$ is called a *homomorphism* if it is a linear map preserving joins and meets, that is, ϕ satisfies

(i) $\phi(f\vee g)=\phi(f)\vee\phi(g)$, $\phi(f\wedge g)=\phi(f)\wedge\phi(g)$, and
(ii) $\phi(\lambda f+\mu g)=\lambda\phi(f)+\mu\phi(g)$

for all $f,g\in\mathcal{L}$, $\lambda,\mu\in\mathbb{R}$. Note that conditions (i) and (ii) implies

(iii) $\phi(|f|)=|\phi(f)|$ for all $f\in\mathcal{L}$.

Indeed, the formulation

$$|f| = f\vee 0 - f\wedge 0$$

implies that

$$\begin{aligned}\phi(|f|) &= \phi(f)\vee\phi(0)-\phi(f)\wedge\phi(0)\\ &= \phi(f)\vee 0-\phi(f)\wedge 0\\ &= |\phi(f)|.\end{aligned}$$

Also, condition (i) follows from conditions (ii) and (iii). To see this, first note that

$$f\vee g=\tfrac{1}{2}(f+g+|f-g|) \text{ and } f\wedge g=\tfrac{1}{2}(f+g-|f-g|).$$

Then we have

$$\begin{aligned}\phi(f\vee g) &= \phi\left(\tfrac{1}{2}(f+g+|f-g|)\right)\\ &= \tfrac{1}{2}\left(\phi(f)+\phi(g)-|\phi(f)-\phi(g)|\right)\\ &= \phi(f)\vee\phi(g).\end{aligned}$$

Similarly, we have $\phi(f\wedge g)=\phi(f)\wedge\phi(g)$. As is well-known, join and meet induce a partial order $\le$ on $H(\mathcal{L})$, that is,

$$f\le g \Longleftrightarrow f=f\wedge g$$

or equivalently,

$$f\le g \Longleftrightarrow g=f\vee g.$$

Then the condition (i) implies that

(iv) $\phi(f)\le\phi(g)$ whenever $f\le g$.

Besides, condition (iii) implies that a homomorphism ϕ is *positive*, that is,

(v) $\phi(f)\ge 0$ whenever $f\in\mathcal{L}$ satisfies $f\ge 0$.

The set of all homomorphisms $\phi:\mathcal{L}\to\mathbb{R}$ is denoted by $H(\mathcal{L})$. Note that $H(\mathcal{L})$ is a subset of $\mathbb{R}^{\mathcal{L}}$. We always consider the topology on $H(\mathcal{L})$ inherited from $\mathbb{R}^{\mathcal{L}}$. Put

$$K(\mathcal{L})=\{\phi\in H(\mathcal{L}):\phi(1)=1\}.$$

Then it is easy to see that $K(\mathcal{L})\subset H(\mathcal{L})$, and $H(\mathcal{L})$ and $K(\mathcal{L})$ are closed subspaces of $\mathbb{R}^{\mathcal{L}}$. In particular, $H(\mathcal{L}^*)$ and $K(\mathcal{L}^*)$ are compact spaces. Indeed, they are closed subspaces of the Cartesian product

$$\prod_{f\in\mathcal{L}^*}\left[\inf f,\sup f\right].$$

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For each $x \in X$, let $\delta_x : \mathcal{L} \to \mathbb{R}$ be the evaluation homomorphism defined by $\delta_x(f) = f(x)$ for every $f \in \mathcal{L}$. We note that $\delta_x(1) = 1$ for every $x \in X$. Then define

$$\delta : X \to K(\mathcal{L})$$

by $\delta(x) = \delta_x$ for each $x \in X$. In case we treat $\mathcal{L}^*$, consider the map

$$e_{\mathcal{L}^*} : X \to \prod_{f \in \mathcal{L}^*} [\inf f, \sup f],$$

defined by $e_{\mathcal{L}^*}(x) = (f(x))$ for every $x \in X$. One should note that two maps $\delta : X \to K(\mathcal{L}^*) \subset \mathbb{R}^{\mathcal{L}}$ and $e_{\mathcal{L}^*} : X \to \prod_{f \in \mathcal{L}^*} [\inf f, \sup f] \subset \mathbb{R}^{\mathcal{L}}$ are essentially the same correspondence.

Recall that a basic neighborhood of $\phi \in H(\mathcal{L})$ (or $\phi \in K(\mathcal{L})$) is given by

$$V(\phi; f_1, \ldots, f_n; \varepsilon) = \{\phi \in \mathcal{L} : |\phi(f_i) - \phi(f_i)| < \varepsilon, \ \forall i = 1, \ldots, n\},$$

where $\varepsilon > 0$ and $f_i \in \mathcal{L}$, $i = 1, \ldots, n$.

Proposition 1 *The evaluation map $\delta : X \to K(\mathcal{L})$ is continuous and $\delta(X)$ is dense in $K(\mathcal{L})$.*

Proof The continuity of δ is trivial. We shall show that $\delta(X)$ is dense in $K(\mathcal{L})$. Suppose the contrary that $\delta(X)$ is not dense in $K(\mathcal{L})$. Then we can take $\phi \in K(\mathcal{L})$ and its basic neighborhood $V(\phi; f_1, \ldots, f_n; \varepsilon)$ of ϕ missing δ_x for every $x \in X$, that is, $|\delta_x(f_i) - \phi(f_i)| \geq \varepsilon$ for every $x \in X$. Consider the map

$$g = \bigvee_{i=1}^{n} \left| f_i - \phi(f_i) \cdot 1 \right|.$$

It is easy to see that $g \in \mathcal{L}$ and $g \geq \varepsilon \cdot 1$. Then we have

$$\phi(g) = \bigvee_{i=1}^{n} \left| \phi(f_i) - \phi(f_i) \cdot \phi(1) \right| = 0.$$

One should note that our assumption $\phi(1) = 1$ is essential to get this equality. On the other hand, we have $\phi(\varepsilon \cdot 1) = \varepsilon \cdot \phi(1) = \varepsilon \cdot 1 > 0$, thus, ϕ cannot be a homomorphism, a contradiction. □

Though $K(\mathcal{L})$ is not compact in general, it can be considered as a realcompactification of X by Proposition 1. See [5] for more information about realcompactifications.

A unital vector lattice $\mathcal{L} \subset C(X)$ is said to *separate points and closed sets* in X provided that, for each close set $F \subset X$ and each point $p \in X \setminus F$, there exists $f \in \mathcal{L}$ such that $f(p) \notin \mathrm{cl}_X F$.

The following is a fundamental fact concerning $K(\mathcal{L})$ (see [5] and [6, 1.7 (j)]).

Proposition 2 *If $\mathcal{L}$ separates points and closed sets in X, then $\delta : X \to K(\mathcal{L})$ is a topological embedding.*

Let $\mathcal{U}(X)$ denote the lattice of all uniformly continuous functions on X. We write $\mathcal{U}$ (resp. $\mathcal{U}^*$) instead of $\mathcal{U}(\mathbb{H})$ (resp. $\mathcal{U}(\mathbb{H})^*$) for notational simplicity. The family $\mathcal{U}^*(X)$ has a ring structure with respect to $\mathbb{R}$, but $\mathcal{U}(X)$ does not. So, we have to consider lattice homomorphisms instead of ring homomorphisms.

Let αX and γX be compactifications of X. We say $\alpha X \geq \gamma X$ provided that there is a continuous map $f : \alpha X \to \gamma X$ such that $f|_X = \mathrm{id}_X$. If $\alpha X \leq \gamma X$ and $\alpha X \geq \gamma X$ then we say that αX and γX are *equivalent compactifications* of X. Of course, two equivalent compactifications of X are homeomorphic.

It is easy to check that $\mathcal{U}^*(X)$ contains all constant maps, separates points from closed sets, and is a closed subring of $C^*(X)$ with respect to the sup-metric, i.e., $\mathcal{U}^*(X)$ is a *complete ring on functions*. Hence, $\mathcal{U}^*(X)$ uniquely determines a compactification uX of X (see [4, 3.12.22 (e)], [6, 4.5]), which is called the *Samuel-Smirnov compactification* of X (see [2], [7]). We note that uX is equivalent to $K(\mathcal{U}^*(X)) = \mathrm{cl}_{\mathbb{R}^{\mathcal{U}^*}} \delta(X)$ because of the equivalence of two maps $\delta : X \to K(\mathcal{U}^*)$ and $e_{\mathcal{U}^*} : X \to \prod_{f \in \mathcal{L}^*} [\inf f, \sup f]$.

The following is a characterization of Samuel-Smirnov compactifications [7, Theorem 2.5].

Theorem 3 *Let αX be a compactification of a metric space X. The following are equivalent:*

- *(a) αX is equivalent to uX.*
- *(b) If $A, B \subset X$ then $cl_{\alpha X} A \cap cl_{\alpha X} B \neq \emptyset$ if and only if $d(A, B) = 0$, where $d(A, B) = \inf\{d(a, b) : a \in A, \ b \in B\}$.*
- *(c) $\{f \in C^*(X) : f$ can be continuously extended to $\alpha X\}$ $= \mathcal{U}^*(X)$.*

Let $Lip(X)$ denote the unital vector lattice of all Lipschitz real-valued functions on X, and $Lip^*(X)$ the vector lattice of all bounded functions in $Lip(X)$. The next theorem gives another description of Samuel-Smirnov compactifications [5, Theorem 3.1].

Theorem 4 *Let (X, d) be a metric space. Then $K(Lip^*(X))$ is equivalent to the Samuel-Smirnov compactification uX of X.*

As a result, we have

$$uX \equiv K(\mathcal{U}^*(X)) \equiv K(Lip^*(X)).$$

The following two results are proved in [2].

Proposition 5 *Let $\phi \in H(\mathcal{U})$. Then ϕ has the form $\phi = c\delta_t$ for some $t \in \mathbb{H}$ and $0 < c < \infty$ if and only if $\phi(1) > 0$. If $\phi(1) = 0$, then $\phi(f) = 0$ for every $f \in \mathcal{U}^*$.*

Corollary 6 $K(\mathcal{U}) = \delta(\mathbb{H})$.

A metric space X is said to be *metrically convex* if, for every two points $x_0, x_1 \in X$ and every $0 < t < 1$, there exists $x_t \in X$ such that $d(x_0, x_t) = td(x_0, x_1)$ and $d(x_1, x_t) = (1 - t)d(x_0, x_1)$. A continuous map $f : X \to Y$ between metric spaces is said to be *Lipschitz for large distance* provided that, for every $\varepsilon > 0$, there exists $L > 0$ depending on ε and f such that

$$d(f(x), f(y)) \leq Ld(x, y)$$

whenever $d(x,y) \geq \varepsilon$.

The following is proved in [1, Proposition 1.11].

Proposition 7 *If X is a metrically convex space, then every uniformly continuous map $f : X \to Y$ from X to a space Y is Lipschitz for large distance.*

Let $\tau : \mathbb{H} \to \mathbb{R}$ be the map defined by

$$\tau(x) = x + 1$$

for every $x \in \mathbb{H}$. By Proposition 7, we have the following:

Corollary 8 *For each $f \in \mathcal{U}$, there is $L > 0$ such that $|f| \leq L\tau$. As a result,* $\limsup_{x\to\infty} x^{-1}|f(x)|$ *is finite. In particular,*

$$\limsup_{t\to\infty} \frac{|f(x)|}{\tau(x)} < \infty.$$

Let $\mathscr{F}$ be an ultrafilter on $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then we define the operation $\lim_{\mathscr{F}(n)}$ by

$$\lim_{\mathscr{F}(n)} f(n) = \bigcap_{F\in\mathscr{F}} \mathrm{cl}\,\{f(n) : n \in F\}$$

for every $f \in C(\mathbb{H})$. Recall that every element of the Stone-Čech compactification $\beta\mathbb{N}_0$ of $\mathbb{N}_0$ can be considered as an ultrafilter on $\mathbb{N}_0$. For each subset $F \subset \mathbb{N}_0$, let $1+F = \{1+x : x \in F\}$. Then $1+\mathscr{F}$ denotes the ultrafilter generated by $\{1+F : F \in \mathscr{F}\}$. Two ultrafilters $\mathscr{F}$ and $1+\mathscr{F}$ are different. In fact, the set of even numbers $2\mathbb{N}$ is contained in only one of $\mathscr{F}$ or $1+\mathscr{F}$.

Put

$$H_\tau = \{\phi \in H(\mathcal{U}) : \phi(\tau) = 1\}.$$

Then define $\tilde{\mu} : [0,1] \times \beta\mathbb{N}_0 \to H_\tau$ by

$$\tilde{\mu}(c,\mathscr{F})(g) = \lim_{\mathscr{F}(n)} \frac{g(2^{c+n}-1)}{2^{c+n}} \quad (g \in \mathcal{U})$$

for every $c \in [0,1]$ and every ultrafilter $\mathscr{F}$ on $\mathbb{N}_0$.

In [2], F. Cabello Sánchez showed that $\tilde{\mu}$ is a continuous surjection. In particular, considering H_τ as the quotient space induced from $\tilde{\mu}$, he proved the following:

Theorem 9 *H_τ is homeomorphic to the quotient obtained from $[0,1] \times \beta\mathbb{N}_0$ after identifying each point of the form $(1,\mathscr{F})$ with $(0, 1+\mathscr{F})$.*

As a corollary, next result follows (see [3]).

Corollary 10 *Each lattice homomorphism $\phi : \mathcal{U} \to \mathbb{R}$ has the form*

$$\phi(g) = \phi(\tau) \cdot \lim_{\mathscr{F}(n)} \frac{g(2^{c+n}-1)}{2^{c+n}} \quad (g \in \mathcal{U})$$

where $\mathscr{F}$ is an ultrafilter on $\mathbb{N}_0$ and $c \in [0,1]$. Moreover, $(c,\mathscr{F})$ and $(d,\mathscr{G})$ induce the same homomorphism if and only if $c = 1,\ d = 0$ and $\mathscr{G} = 1+\mathscr{F}$, or vice-versa.

Next we consider the map $\mu^* : [0,1] \times \mathbb{N}_0 \to K(\mathcal{U}^*)$ defined by $\mu^*(c,n) = \delta_{c+n}$ for every $(c,n) \in [0,1] \times \mathbb{N}_0$. Because each $\phi \in H(\mathcal{U}^*)$ has a basic neighborhood $V(\phi; f_1, \ldots, f_n; \varepsilon)$ for some $\varepsilon > 0$ and $f_i \in \mathcal{U}^*$, $i = 1, \ldots, n$, it follows that μ^* is continuous.

Consider the map $\tilde{\mu}^* : [0,1] \times \beta\mathbb{N}_0 \to K(\mathcal{U}^*)$ defined by

$$\tilde{\mu}^*(c,\mathscr{F})(f) = \lim_{\mathscr{F}(n)} f(c+n) \quad (f \in \mathcal{U}^*)$$

for every $c \in [0,1]$ and every ultrafilter $\mathscr{F}$ on $\mathbb{N}_0$. Since each point $n \in \mathbb{N}_0$ is considered as a limit of a fixed ultrafilter on $\mathbb{N}_0$, the map $\tilde{\mu}^*$ can be considered as the extension of $\mu^* : [0,1] \times \mathbb{N}_0 \to K(\mathcal{U}^*)$. Hence, if we show that $\tilde{\mu}^*$ is continuous, then $\tilde{\mu}^*$ is the unique continuous extension of μ^* and is surjective by the density of the image of μ^*. To see the continuity of $\tilde{\mu}^*$, we remember the topology of $\beta\mathbb{N}_0$. As is well-known, $\beta\mathbb{N}_0$ is identified with the closure of the following evaluation map

$$e_{C^*(\mathbb{N}_0)} : \mathbb{N}_0 \to \prod_{f\in C^*(\mathbb{N}_0)} \left[\inf f, \sup f\right].$$

Then for an ultrafilter $\mathscr{F} \in \beta\mathbb{N}_0$, the corresponding point is represented by

$$\left(\lim_{\mathscr{F}(n)} f(n)\right)_{f\in C^*(\mathbb{N}_0)} \in \prod_{f\in C^*(\mathbb{N}_0)} \left[\inf f, \sup f\right].$$

We put $\phi = \tilde{\mu}^*(c,\mathscr{F})$ and consider a basic neighborhood $V(\phi; f_1, \ldots, f_k; \varepsilon)$ of ϕ in $K(\mathcal{U}^*)$. Then we can take a neighborhood V of $\mathscr{F}$ in $\beta\mathbb{N}_0$ as follows:

$$V = \{\mathscr{G} : |\lim_{\mathscr{G}(n)} f_i(c+n) - \lim_{\mathscr{F}(n)} f_i(c+n)| < \tfrac{\varepsilon}{2},\ 1 \leq \forall i \leq k\}.$$

Using the uniformity of f_i's, we can take a neighborhood U of c in $[0,1]$ such that, $d \in U$ implies that

$$\left|f_i(d+n) - f_i(c+n)\right| < \tfrac{\varepsilon}{2}$$

for every $n \in \mathbb{N}_0$ and every $i = 1, \ldots, k$. Then it follows that $\tilde{\mu}^*(U \times V) \subset V(\phi; f_1, \ldots, f_k; \varepsilon)$. Thus, $\tilde{\mu}^*$ is continuous.

We are going to consider the kernel of $\tilde{\mu}^* : [0,1] \times \beta\mathbb{N}_0 \to K(\mathcal{U}^*)$. Obviously, $\tilde{\mu}^*(c,\mathscr{F}) = \tilde{\mu}^*(d,\mathscr{G})$ when $c = 1,\ d = 0$ and $\mathscr{G} = 1+\mathscr{F}$, or vice-versa. We shall show that $\tilde{\mu}^*(c,\mathscr{F}) \neq \tilde{\mu}^*(d,\mathscr{G})$ in the other cases. We may assume without loss of generality that $0 \leq c \leq d < 1$.

Suppose that $\mathscr{F} \neq \mathscr{G}$. Then there exists $A \subset \mathbb{N}_0$ such that $A \in \mathscr{F}$ but $A \notin \mathscr{G}$ since they are ultrafilters. Define $f : \mathbb{H} \to [0,1]$ as a piecewise linear map such that

$$f(t) = \begin{cases} 1 & \text{if } t = c+n \text{ and } n \in A, \\ 0 & \text{if } t = d+n \text{ and } n \notin A. \end{cases}$$

We can take f as a bounded uniformly continuous map. Then we have

$$\tilde{\mu}^*(c,\mathscr{F})(f) = \lim_{\mathscr{F}(n)} f(c+n) = \lim_{A(n)} f(c+n) = 1$$

but $\tilde{\mu}^*(d,\mathscr{G})(f) = 0$.

If $\mathscr{F} = \mathscr{G}$ but $c \neq d$, then we take a piecewise linear map $g : \mathbb{H} \to [0,1]$ such that

$$g(t) = \begin{cases} 1 & \text{if } t = c + n, \\ 0 & \text{if } t = d + n. \end{cases}$$

Also, we can take g as a bounded uniformly continuous map. Then it is easy to see that $\tilde{\mu}^*(c, \mathscr{F})(g) = 1$ and $\tilde{\mu}^*(d, \mathscr{F})(g) = 0$. Thus, $\tilde{\mu}^*(c, \mathscr{F}) = \tilde{\mu}^*(d, \mathscr{G})$ if and only if $c = 1,\ d = 0$ and $\mathscr{G} = 1 + \mathscr{F}$, or vice-versa.

These estimation of the kernel of $\tilde{\mu}^*$ gives the following, they are in fact stated in [3].

Theorem 11 *The Samuel-Smirnov compactification $K(\mathcal{U}^*)$ of $\mathbb{H}$ is homeomorphic to the quotient obtained from $[0,1] \times \beta\mathbb{N}_0$ after identifying each point of the form $(1, \mathscr{F})$ with $(0, 1 + \mathscr{F})$.*

Corollary 12 *Each lattice homomorphism $\phi : \mathcal{U}^* \to \mathbb{R}$ has the form*

$$\phi(f) = \phi(1) \cdot \lim_{\mathscr{F}(n)} f(c+n) \quad (f \in \mathcal{U}^*)$$

where $\mathscr{F}$ is an ultrafilter on $\mathbb{N}_0$ and $c \in [0,1]$. Moreover, $(c, \mathscr{F})$ and $(d, \mathscr{G})$ induce the same homomorphism if and only If $c = 1,\ d = 0$ and $\mathscr{G} = 1 + \mathscr{F}$, or vice-versa.

As F. Cabello Sánchez said in [3], Corollary 12 can be considered as a description of Samuel-Smirnov compactification of the half-line when we restrict to $K(\mathcal{U}^*)$.

Let $C_0(X) \subset C(X)$ denote the set of functions which are *small off compact sets*, that is, $f \in C_0(X)$ if and only if, given $\varepsilon > 0$, there exists a compact subset K of X such that $|f(x)| < \varepsilon$ for every $x \in X \setminus K$. We write C_0 instead of $C_0(\mathbb{H})$ for notational simplicity. It is easy to see that each element of $C_0(X)$ is uniformly continuous and bounded, i.e., $C_0(X) \subset \mathcal{U}^*(X)$. Also, $C_0(X)$ is an ideal of $\mathcal{U}^*(X)$. So, we can consider the quotient $\mathcal{U}^*(X)/C_0(X)$. For each $f \in \mathcal{U}^*(X)$, $[f]$ denotes the equivalence class of f. We define join and meet on $\mathcal{U}^*(X)/C_0(X)$ by $[f] \vee [g] = [f \vee g]$ and $[f] \wedge [g] = [f \wedge g]$. The well-definedness of these follows easily. Indeed, each $f' \in [f]$ and $g' \in [g]$ can be expressed as $f' = f + h_1$ and $g' = g + h_2$ for some $h_1,\ h_2 \in C_0(X)$. Note that we have $|(f+h) \vee g - f \vee g| \leq |h|$ for every $f, g, h \in C(X)$. Then this inequality implies that

$$\begin{aligned} &|(f+h_1) \vee (g+h_2) - f \vee g| \\ &\quad \leq |(f+h_1) \vee (g+h_2) - f \vee (g+h_2)| \\ &\qquad + |f \vee (g+h_2) - f \vee g| \leq |h_1| + |h_2|. \end{aligned}$$

This means that $(f+h_1) \vee (g+h_2) - f \vee g \in C_0(X)$. Hence, we have $[f] \vee [g] = [f'] \vee [g']$ for every $f' \in [f]$ and $g' \in [g]$. Similarly, we have $[f] \wedge [g] = [f'] \wedge [g']$ for every $f' \in [f]$ and $g' \in [g]$.

In what follows, X is assumed to be a proper metric space. Then the remainder $uX \setminus X$ of Samuel-Smirnov compactification is compact. It should be remarked that $C(uX) = C^*(uX) = \mathcal{U}^*(X)$, and $C(uX \setminus X) = C^*(uX \setminus X)$, in particular, $C(uX \setminus X)$ is isomorphic to $\mathcal{U}^*(X)/C_0(X)$. Thus, the homomorphisms on $\mathcal{U}^*(X)/C_0(X)$ can be considered as homomorphisms of the continuous functions on the remainder of Samuel-Smirnov compactification uX of X, that is,

$$H(\mathcal{U}^*(X)/C_0(X)) \equiv H(C(uX \setminus X)).$$

As we have already seen, $K(C^*(uX \setminus X))$ is a compact space containing $\delta(uX \setminus X)$ as a dense subspace. Since $\delta(uX \setminus X)$ is compact, $K(C^*(uX \setminus X))$ is equivalent to $\delta(uX \setminus X)$, that is, $K(C^*(uX \setminus X))$ is homeomorphic to the remainder of Samuel-Smirnov compactification of X, that is,

$$K(\mathcal{U}^*(X)/C_0(X)) \equiv uX \setminus X.$$

Let $\bar{\phi} \in H(\mathcal{U}^*(X)/C_0(X))$. If we define $\phi(f) = \bar{\phi}([f])$ for every $f \in \mathcal{U}^*(X)$, then ϕ becomes a homomorphism on $\mathcal{U}^*(X)$ with the property that $\phi(h) = 0$ for every $h \in C_0(X)$. Conversely, if $\phi \in H(\mathcal{U}^*(X))$ satisfies $\phi(h) = 0$ for every $h \in C_0(X)$, then the map $\bar{\phi} : \mathcal{U}^*(X)/C_0(X) \to \mathbb{R}$ defined by $\bar{\phi}([f]) = \phi(f)$ is well-defined and becomes a homomorphism. Thus, we can identify each homomorphism $\bar{\phi} \in H(\mathcal{U}^*(X)/C_0(X))$ with a homomorphism $\phi \in H(\mathcal{U}^*(X))$ with $\phi|_{C_0(X)} = 0$.

Now we consider the remainder $u\mathbb{H} \setminus \mathbb{H}$ of the Samuel-Smirnov compactification of the half-line, that is, the homomorphisms on $\mathcal{U}^*/C_0$. Let $\phi \in H(\mathcal{U}^*)$ be such that $\phi|_{C_0} = 0$. By Corollary 12, we can take an ultrafilter $\mathscr{F}$ on $\mathbb{N}_0$ and $c \in [0,1]$ such that

$$\phi(f) = \phi(1) \cdot \lim_{\mathscr{F}(n)} f(c+n)$$

for every $f \in \mathcal{U}^*$. If $\mathscr{F}$ is a fixed ultrafilter, then $\phi = \phi(1) \cdot \delta_{c+x}$ where x is the limit point of $\mathscr{F}$. Therefore, ϕ cannot be zero on C_0. If $\mathscr{F}$ is a free ultrafilter on $\mathbb{N}_0$, then

$$\lim_{\mathscr{F}(n)} h(c+n) = 0$$

for every $h \in C_0$. Thus, these arguments give a lattice theoretic proof of the following structure theorem, which is a one-ended version of [7, Theorem 4.8]:

Theorem 13 *The remainder $u\mathbb{H} \setminus \mathbb{H}$ of the Samuel-Smirnov compactification of the half-line is homeomorphic to the quotient obtained from $[0,1] \times (\beta\mathbb{N}_0 \setminus \mathbb{N}_0)$ after identifying each point of the form $(1, \mathscr{F})$ with $(0, 1 + \mathscr{F})$.*

Corollary 14 *Each lattice homomorphism $\phi : \mathcal{U}^*/C_0 \to \mathbb{R}$ has the form*

$$\phi([f]) = \phi(1) \cdot \lim_{\mathscr{F}(n)} f(c+n) \quad (f \in \mathcal{U}^*)$$

where $\mathscr{F}$ is an free ultrafilter on $\mathbb{N}_0$ and $c \in [0,1]$. Moreover, $(c, \mathscr{F})$ and $(d, \mathscr{G})$ induce the same homomorphism if and only If $c = 1,\ d = 0$ and $\mathscr{G} = 1 + \mathscr{F}$, or vice-versa.

Remark 15 In [7, Theorem 4.8], Woods proved that the remainder $u\mathbb{R} \setminus \mathbb{R}$ can be written as a union of two copies of

$[0, 1] \times (\beta\omega \setminus \omega)$ (where ω is the countably infinite discrete space), and that their intersection is a nowhere dense copy of $\beta\omega \setminus \omega$. Though we adopted $\mathbb{N}_0$ to describe the formula, it is of course homeomorphic to ω. Identification of two points $(1, \mathscr{F})$ with $(0, 1 + \mathscr{F})$ given in Theorem 13 runs all over the free ultrafilters of $\mathbb{N}_0$. Thus, Theorem 13 provides a concrete description of one end version of those given in [7].

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