Lower lines of the $E(3)$-based Adams E_2-term for $M(5, v_1, v_2^2)$

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Let $E(3)$ be the third Johnson-Wilson spectrum at the prime five. In this note, we show that the generalized Smith-Toda spectrum $V_2 = M(5, v_1, v_2^2)$ is a ring spectrum. Furthermore, we consider E_2-terms of the $E(3)$-based Adams spectra sequence converging to the homotopy groups of the $E(3)$-localization of V_2.

1. Introduction

For a prime number p, we have the Brown-Peterson homology theory $BP_*(-)$ at p, whose coefficient algebra is

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \dots] \text{ with } |v_i| = 2(p^i - 1).$$

For an invariant regular ideal $J_k = (p^{e_0}, v_1^{e_1}, \dots, v_{k-1}^{e_{k-1}})$ of BP_*, the generalized Smith-Toda spectrum MJ_k is defined by

$$BP_*(MJ_k) = BP_*/J_k.$$

We know that the spectrum $M(p, v_1, v_2^k)$ for $p \geq 5$ and $k \geq 1$ exists. At $p = 5$, Ravenel showed that $M(5, v_1, v_2)$ has no ring spectrum structure. The first main theorem in this note is the following:

Theorem 1 The spectrum $V_2 = M(5, v_1, v_2^2)$ is a ring spectrum.

For $n > 0$, we denote

$$E(n)_* = v_n^{-1}BP_*/(v_{n+1}, v_{n+2}, \dots) = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_{n-1}, v_n^{\pm 1}].$$

The nth Johnson-Wilson spectrum $E(n)$ is a spectrum which represents the homology theory

$$E(n)_*(-) = E(n)_* \otimes_{BP_*} BP_*(-).$$

If $k \leq n$, then we may consider that J_k is an invariant regular ideal of $E(n)_*$, and the spectrum MJ_k satisfies

$$E(n)_*(MJ_k) = E(n)_* \otimes_{BP_*} BP_*(MJ_k) = E(n)_*/J_k.$$

Let L_n denote the Bousfield localization functor with respect to $E(n)$. The $E(n)$-based Adams spectral sequence for a spectrum X is of the form

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*, E(n)_*(X)) \Rightarrow \pi_{t-s}(L_n X).$$

Hereafter, we denote by $E(n)_r^{s,t}(X)$ the E_r-term of the spectral sequence.

We put $k(n-1)_* = \mathbb{Z}/p[v_{n-1}]$ and

$$E_i(n, m)_* = \mathbb{Z}/p[v_{n-1}, v_n^{\pm p^m}]/(v_{n-1}^i) \qquad (2)$$

for $i \geq 1$, $m \geq 0$ and $n \geq 2$. The second main theorem in this note is the following:

Theorem 3 Assume $p = 5$. As a $k(2)_*$-module, $E(3)_2^{s,*}(V_2)$ for $s \leq 2$ is the following:

$$\begin{aligned}
E(3)_2^{0,*}(V_2) &= E_1(3,1)_*\{v_2 v_3^t \colon t = 1,2,3,4\} \oplus E_2(3,1)_*,\\
E(3)_2^{1,*}(V_2) &= E_1(3,1)_*\{v_3^{t-1}h_2, v_2 v_3^t \zeta_3 \colon t = 1,2,3,4\}\\
&\quad \oplus E_2(3,0)_*\{h_0, h_1\} \oplus E_2(3,1)_*\{v_3^{-1}h_2, \zeta_3\},\\
E(3)_2^{2,*}(V_2) &= E_1(3,1)_*\{v_3^{t-1}\zeta_3, v_2 v_3^t b_i, v_2 v_3^t g_i, v_2 v_3^t k_i \colon i = 0,1,2,\ t = 1,2,3,4\}\\
&\quad \oplus E_2(3,0)_*\{h_0\zeta_3, h_1\zeta_3\} \oplus E_2(3,1)_*\{b_i, g_i, k_i, v_3^{-1}\zeta_3 \colon i = 0,1,2\}.
\end{aligned}$$

Here, the generators have the following internal degrees:

$$|h_i| = 8 \cdot 5^i, \quad |\zeta_3| = 0, \quad |\zeta_3 h_i| = 8 \cdot 5^i,$$
$$|b_i| = 8 \cdot 5^{i+1}, \quad |g_i| = 56 \cdot 5^i \quad \text{and} \quad |k_i| = 88 \cdot 5^i.$$

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2. The ring structure on $V_2 = M(5, v_1, v_2^2)$

We begin with a little stronger theorem than [1, Th. 1].

Theorem 4 Let X be a ring spectrum with a multiplication $\mu \colon X \wedge X \to X$, and let $f \colon \Sigma^{|f|}X \to X$ with $|f|$ even. Suppose that:

1. The composite

$$\Sigma^{|f|}X \wedge C(f) \xrightarrow{f \wedge 1_{C(f)}} X \wedge C(f) \xrightarrow{m} C(f)$$

is trivial. Here, $C(f)$ is the cofiber of f and the map m is the X-module structure on $C(f)$ obtained from the next condition.

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2. The diagram

$$\begin{array}{ccc} \Sigma^{|f|}X\wedge X & \xrightarrow{1_X\wedge f} & X\wedge X \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ \Sigma^{|f|}X & \xrightarrow{f} & X \end{array}$$

commutes.

Then, $C(f)$ has a ring structure so that the inclusion map $\iota\colon X\to C(f)$ is a map of ring spectra.

Proof. The proof is standard and identical to that of [1, Th. 1]. Let $\eta\colon S^0\to X$ be the unit map on X. The hypothesis (2) implies the existence of a map $m'\colon X\wedge C(f)\to C(f)$ fitting in the commutative diagram

$$\begin{array}{ccccccc} \Sigma^{|f|}S^0\wedge X & \xrightarrow{1_{S^0}\wedge f} & S^0\wedge X & \longrightarrow & S^0\wedge C(f) & \longrightarrow & \Sigma^{|f|+1}S^0\wedge X \\ {\scriptstyle\eta\wedge 1_X}\downarrow & & \downarrow{\scriptstyle\eta\wedge 1_X} & & \downarrow{\scriptstyle\eta\wedge 1_{C(f)}} & & \downarrow{\scriptstyle\eta\wedge 1_X} \\ \Sigma^{|f|}X\wedge X & \xrightarrow{1_X\wedge f} & X\wedge X & \longrightarrow & X\wedge C(f) & \longrightarrow & \Sigma^{|f|+1}X\wedge X \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle m'} & & \downarrow{\scriptstyle\mu} \\ \Sigma^{|f|}X & \xrightarrow{f} & X & \longrightarrow & C(f) & \longrightarrow & \Sigma^{|f|+1}X, \end{array}$$

in which the rows are cofiber sequences. Since $m'(\eta\wedge 1_{C(f)})$ is an automorphism of $C(f)$, we put $m=(m'(\eta\wedge 1_{C(f)}))^{-1}m'$. Then, $m(\eta\wedge 1_{C(f)})=1_{C(f)}$.

By the hypothesis (1), we have a map $\mu'\colon C(f)\wedge C(f)\to C(f)$ such that $\mu'(\iota\wedge 1_{C(f)})=m$. Put also $\eta'=\iota\eta\colon S^0\to C(f)$ for the inclusion $\iota\colon X\to C(f)$. Then, we have a commutative diagram

$$\begin{array}{ccccc} * & \longrightarrow & S^0\wedge C(f) & = & S^0\wedge C(f) \\ \downarrow & & \downarrow{\scriptstyle\eta\wedge 1_{C(f)}} & & \downarrow{\scriptstyle\eta'\wedge 1_{C(f)}} \\ \Sigma^{2|f|}X\wedge C(f) & \xrightarrow{f\wedge 1_{C(f)}} & X\wedge C(f) & \xrightarrow{\iota\wedge 1_{C(f)}} & C(f)\wedge C(f) \\ \downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle\mu'} \\ * & \longrightarrow & C(f) & = & C(f), \end{array}$$

Therefore, we obtain the ring structure map on $C(f)$ so that ι is a map of ring spectra. □

From now on, we work at the prime five. We consider the mod five Moore spectrum M and the first Smith-Toda spectrum $V(1)$ defined by the cofiber sequences

$$S^0\xrightarrow{5}S^0\xrightarrow{i}M\xrightarrow{j}S^1 \text{ and } \Sigma^8M\xrightarrow{\alpha}M\xrightarrow{i_1}V(1)\xrightarrow{j_1}\Sigma^9M$$

for the Adams map α. For X and f in Theorem 4, we set

$$X=V(1) \quad\text{and}\quad f=\beta\colon\Sigma^{48}V(1)\to V(1).$$

Here, β is the v_2-periodic map due to L. Smith. We notice that $V(1)$ is a commutative associative ring spectrum (*cf.* [2, Remark 3.9]), and β satisfies the condition (2) of Theorem 4 by [2, p. 41]. Consider the cofiber sequences

(5) $$\Sigma^{48t}V(1)\xrightarrow{\beta^t}V(1)\xrightarrow{i'_t}V_t\xrightarrow{j'_t}\Sigma^{48t+1}V(1).$$

Then, the cofiber $C(\beta^2)$ is $V_2=M(5,v_1,v_2^2)$ and we have the $V(1)$-module structure $m\colon V(1)\wedge V_2\to V_2$ on V_2 obtained from the condition (2) so that

(6) $$i'_2\mu_1=m(1_{V(1)}\wedge i'_2),$$

where $\mu_1\colon V(1)\wedge V(1)\to V(1)$ denotes the multiplication of the ring spectrum $V(1)$. Furthermore, Verdier's axiom gives rise to a cofiber sequence

(7) $$\Sigma^{48}V_1\xrightarrow{\lambda}V_2\xrightarrow{\rho}V_1\to\Sigma^{49}V_1$$

from the cofiber sequences (5) for $t\in\{1,2\}$ so that

(8) $$\lambda i'_1=i'_2\beta \quad\text{and}\quad \rho i'_2=i'_1.$$

Lemma 9 $[V(1)\wedge V_2,V_2]_{48}=\mathbb{Z}/5\left\{m(1_{V(1)}\wedge\lambda\rho)\right\}$.

Proof. From [5, Th. 3.6], for degree <187, we obtain

(10) $$[V(1),V_2]_*\cong P(\beta,\beta')/(\beta^2)\otimes A\otimes E(\delta_0)\oplus P(\beta,\beta')/(\beta^2)\otimes B,$$

where $|\beta'|=38$, $|\delta_0|=-10$, A is a $\mathbb{Z}/5$-module generated by six elements of degrees

$$0,\quad 7,\quad 39,\quad 54,\quad 86\quad\text{and}\quad 93,$$

and B is a $\mathbb{Z}/5$-module generated by eight elements of degrees

$$-9,\quad 6,\quad 30,\quad 38,\quad 45,\quad 53,\quad 77\quad\text{and}\quad 92.$$

In [2, Remark 3.9], it is shown

$$V(1)\wedge V(1)=V(1)\vee(\Sigma L_1\wedge V(1))\vee\Sigma^{10}V(1)$$

for the cofiber L_1 of $\alpha_1\in\pi_7(S^0)$. ([2, Remark 3.9] has a typo: $sp+2$ is actually $sq+2$.) By this decomposition together with (10), we obtain

$$[V(1)\wedge V(1),V_2]_{145}=0 \quad\text{and}\quad [V(1)\wedge V(1),V_2]_{48}=\mathbb{Z}/5\{i'_2\beta\mu_1\}.$$

It follows that i'_2 induces the monomorphism

$$(1_{V(1)}\wedge i'_2)^*\colon [V(1)\wedge V_2,V_2]_{48}\to[V(1)\wedge V(1),V_2]_{48}=\mathbb{Z}/5\{i'_2\beta\mu_1\}.$$

For the element $m(1_{V(1)}\wedge\lambda\rho)\in[V(1)\wedge V_2,V_2]_{48}$, we compute by (6) and (8):

$$\begin{aligned}(1_{V(1)}\wedge i'_2)^*(m(1_{V(1)}\wedge\lambda\rho)) &= m(1_{V(1)}\wedge\lambda\rho)(1_{V(1)}\wedge i'_2)=m(1_{V(1)}\wedge\lambda\rho i'_2)\\ &= m(1_{V(1)}\wedge\lambda i'_1)=m(1_{V(1)}\wedge i'_2\beta)\\ &= m(1_{V(1)}\wedge i'_2)(1_{V(1)}\wedge\beta)=i'_2\mu_1(1_{V(1)}\wedge\beta)\\ &= i'_2\beta\mu_1.\end{aligned}$$

Thus, we see that $(1_{V(1)}\wedge i'_2)^*$ is an isomorphism and the lemma follows. □

Proof of Theorem 1. It suffices to show that $X=V(1)$ and $f=\beta$ satisfy the condition (1) in Theorem 4. Consider the element $m(\beta\wedge 1_{V_2})\in[V(1)\wedge V_2,V_2]_{48}$. Then, by Lemma 9, there exists $x\in\mathbb{Z}/5$ such that

$$m(\beta\wedge 1_{V_2})=xm(1_{V(1)}\wedge\lambda\rho).$$

Now we verify the condition (1) by computation

$$\begin{aligned} m(\beta^2 \wedge 1_{V_2}) &= m(\beta \wedge 1_{V_2})(\beta \wedge 1_{V_2}) \\ &= xm(1_{V(1)} \wedge \lambda\rho)(\beta \wedge 1_{V_2}) \\ &= xm(\beta \wedge 1_{V_2})(1_{V(1)} \wedge \lambda\rho) \\ &= x^2 m(1_{V(1)} \wedge \lambda\rho)(1_{V(1)} \wedge \lambda\rho) \\ &= x^2 m(1_{V(1)} \wedge \lambda\rho\lambda\rho) \\ &= 0, \end{aligned}$$

since $\rho\lambda = 0$ by (7). □

3. $E(3)_2^{s,*}(M(5, v_1, v_2))$ for $s \le 2$

Let $E(3)$ be the third Johnson-Wilson spectrum at $p = 5$. For an $E(3)_*(E(3))$-comodule M, we denote

$$H^s M = \mathrm{Ext}^{s,*}_{E(3)_*(E(3))}(E(3)_*, M).$$

Put $K(3)_* = E(3)_*/(5, v_1, v_2) = \mathbb{Z}/5[v_3^{\pm 1}]$, and we have

$$H^s K(3)_* = E(3)_2^{s,*}(M(5, v_1, v_2)).$$

From [3, Th. 6.3.34], we obtain the structure of $H^s M_3^0$. In particular,

$$\begin{aligned} H^0 K(3)_* &= K(3)_*, \\ H^1 K(3)_* &= K(3)_*\{h_i, \zeta_3 \colon i = 0, 1, 2\}, \\ H^2 K(3)_* &= K(3)_*\{h_i\zeta_3, b_i, g_i, k_i \colon i = 0, 1, 2\}. \end{aligned}$$

(Remark that Ravenel used $h_{1,i}$ and $b_{1,i}$ instead of h_i and b_i, respectively.) We also note that [4, 2.7 Th.] implies that $H^3 K(3)_*$ contains a subspace generated by

$$h_i b_i \text{ and } h_i b_{i+2} \quad \text{for } i \in \mathbb{Z}/3.$$

By the argument of [4, p.954], in $H^3 S(3)_*$, we have

$$h_i b_{i+1} = h_i \langle h_{i+1}, h_{i+2}, h_i, h_{i+1}\rangle = \langle h_i, h_{i+1}, h_{i+2}, h_i\rangle h_{i+1} = b_i h_{i+1}.$$

For $E(3)_* = \mathbb{Z}_{(5)}[v_1, v_2, v_3^{\pm 1}]$, we consider the subquotient

$$E_i(3, m)_* = \mathbb{Z}/5[v_2, v_3^{\pm 5^m}]/(v_2^i).$$

In particular, $K(3)_* = E_1(3, 0)_*$. Furthermore,

$$H^s E_2(3, 0)_* = E(3)_2^{s,*}(V_2)$$

for $V_2 = M(5, v_1, v_2^2)$. The short exact sequence

$$0 \to K(3)_* \xrightarrow{v_2} E_2(3, 0)_* \to K(3)_* \to 0$$

gives rise to the connecting homomorphism

$$\partial_s \colon H^s K(3)_* \to H^{s+1} K(3)_*,$$

It is easy to see the following lemma:

Lemma 11 Let $t \in \mathbb{Z}$.

$$\begin{aligned} \partial_0(v_3^t) &= \begin{cases} t v_3^{t-1} h_2 & 5 \nmid t, \\ 0 & 5 \mid t, \end{cases} \\ \partial_1(v_3^t h_i) &= 0 \quad \text{for any } t, \\ \partial_1(v_3^t \zeta_3) &= \begin{cases} t v_3^{t-1} h_2 \zeta_3 & 5 \nmid t, \\ 0 & 5 \mid t, \end{cases} \\ \partial_2(v_3^t h_i \zeta_3) &= 0 \quad \text{for any } t, \\ \partial_2(v_3^t b_i) &= \begin{cases} t v_3^{t-1} h_2 b_i & 5 \nmid t, \\ 0 & 5 \mid t, \end{cases} \\ \partial_2(v_3^t g_i) &= \begin{cases} t v_3^{t-1} h_2 g_1 & t \nmid 5 \text{ and } i = 1, \\ 0 & \text{otherwise} \end{cases} \\ \partial_2(v_3^t k_i) &= \begin{cases} t v_3^{t-1} h_2 k_2 \left(= t v_3^{t-1} g_2 h_0\right) & t \nmid 5 \text{ and } i = 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof of Theorem 3. Consider the exact sequence

$$H^{s-1} K(3)_* \xrightarrow{\partial_{s-1}} H^s K(3)_* \xrightarrow{(v_2)_*} H^s E_2(3, 0)_* \to H^s K(3)_* \xrightarrow{\partial_s} H^{s+1} K(3)_*.$$

From Lemma 11, we obtain

$$\begin{aligned} H^0 K(3)_* &= K(3)_*, \\ \mathrm{Ker}(\partial_0) &= \mathbb{Z}/5[v_3^{\pm 5}], \\ \mathrm{Coker}(\partial_0) &= K(3)_*\{h_0, h_1, \zeta_3\} \oplus \mathbb{Z}/5[v_3^{\pm 5}]\{v_3^{-1} h_2\}, \\ \mathrm{Ker}(\partial_1) &= K(3)_*\{h_i \colon i = 0, 1, 2\} \oplus \mathbb{Z}/5[v_3^{\pm 5}]\{\zeta_3\}, \\ \mathrm{Coker}(\partial_1) &= K(3)_*\{h_0\zeta_3, h_1\zeta_3, b_i, g_i, k_i \colon i \in \mathbb{Z}/3\} \oplus \mathbb{Z}/5[v_3^{\pm 5}]\{v_3^{-1} h_2 \zeta_3\}, \\ \mathrm{Ker}(\partial_2) &= K(3)_*\{h_i\zeta_3 \colon i = 0, 1, 2\} \oplus \mathbb{Z}/5[v_3^{\pm 5}]\{b_i, g_i, k_i \colon i = 0, 1, 2\}. \end{aligned}$$

We have the isomorphism

$$H^s E_2(3, 0)_* = v_2 \mathrm{Coker}(\partial_{s-1}) \oplus \mathrm{Ker}(\partial_s),$$

of $\mathbb{Z}/5$-vector spaces. Therefore

$$\begin{aligned} H^0 E_2(3, 0)_* &= E_1(3, 1)_*\{v_2 v_3^t \colon t = 1, 2, 3, 4\} \oplus E_2(3, 1)_*, \\ H^1 E_2(3, 0)_* &= E_1(3, 1)_*\{v_3^{t-1} h_2, v_2 v_3^t \zeta_3 \colon t = 1, 2, 3, 4\} \\ &\quad \oplus E_2(3, 0)_*\{h_0, h_1\} \oplus E_2(3, 1)_*\{v_3^{-1} h_2, \zeta_3\}, \\ H^2 E_2(3, 0)_* &= E_1(3, 1)_*\{v_3^{t-1} \zeta_3, v_2 v_3^t b_i, v_2 v_3^t g_i, v_2 v_3^t k_i \colon i = 0, 1, 2,\ t = 1, 2, 3, 4\} \\ &\quad \oplus E_2(3, 0)_*\{h_0\zeta_3, h_1\zeta_3\} \oplus E_2(3, 1)_*\{b_i, g_i, k_i, v_3^{-1}\zeta_3 \colon i = 0, 1, 2\}. \end{aligned}$$

□

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