Lower lines of the $E(2)$-based Adams spectral sequence for $M(2, v_1^4)$

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In this note, we calculate the zeroth, first, second and third lines of the $E(2)$-based Adams spectral sequence for the type two finite spectrum $M(2, v_1^4)$.

1. Introduction

Consider the homology theory $BP_*(-)$ represeted by the Brown-Peterson spectrum BP at 2. The coefficient ring of $BP_*(-)$ is $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, v_3, \dots]$ with $|v_i| = 2^{i+1} - 2$. In the stable homotopy category of 2-localized spectra, there exists a type two finite spectrum $V_4 = M(2, v_1^4)$, whose BP_*-homology is

$$BP_*(V_4) = BP_*/(2, v_1^4).$$

The n-th Johnson-Wilson theory $E(n)_*(-)$ at 2 is defined by the homology theory

$$E(n)_*(-) = E(n)_* \otimes_{BP_*} BP_*(-),$$

where

$$E(n)_* = v_n^{-1}BP_*/(v_{n+1}, v_{n+2}, \dots) = \mathbb{Z}_{(2)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}].$$

The spectrum $E(n)$ is defined to be the spectrum which represents $E(n)_*(-)$. We then have

$$E(n)_*(V_4) = E(n)_* \otimes_{BP_*} BP_*(V_4) = E(n)_*/(2, v_1^4).$$

We denote by L_n the Bousfield localization functor with respect to $E(n)$. The $E(n)$-based Adams spectral sequence for a spectrum X is of the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*, E(n)_*(X)) \Rightarrow \pi_{t-s}(L_n X).$$

We denote by $E(n)_r^{s,t}(X)$ the E_r-term of the spectral sequence. Our aim in this note is to calculate $E(2)_2^s(V_4)$ for $s \le 3$.

Hereafter, for an $E(2)_*(E(2))$-comodule M, we denote

$$H^s M = \mathrm{Ext}^{s,*}_{E(2)_*(E(2))}(E(2)_*, M).$$

Consider the following $E(2)_*(E(2))$-comodules:

$$N_1^0 = E(2)_*/(2), \quad M_1^0 = v_1^{-1}N_1^0,$$
$$M_1^1 = \mathrm{Coker}(N_1^0 \xrightarrow{\subset} M_1^0) = E(2)_*/(2, v_1^\infty),$$
$$K(2)_* = E(2)_*/(2, v_1) = \mathbb{Z}/2[v_2^{\pm 1}].$$

We also put

$$k(1)_* = BP_*/(2, v_2, v_3, \dots) = \mathbb{Z}/2[v_1],$$
$$K(1)_* = E(1)_*/(2) = \mathbb{Z}/2[v_1^{\pm 1}].$$

Consider the canonical projection $\tau\colon E(2)_*(V_4) \to M_2^0$, and, for $x \in H^* M_2^0$, the notation $x_s \in E(2)_2^{*,*}(V_4) = H^*(E(n)_*/(2, v_1^4))$ for $s \in \mathbb{Z}$ is defined by $\tau_*(x_s) = v_2^s x \in H^* M_2^0$. We put $k_i(1)_* = \mathbb{Z}/2[v_1]/(v_1^i)$ for $i \ge 1$.

Theorem 1 As a $k(1)_*$-module, the E_2-term $E(2)_2^{s,*}(V_4)$ for $s \le 3$ is given as follows:

- $E(2)_2^{0,*}(V_4)$ is a direct sum of
$$k_1(1)_*\{v_1^3 1_{2t+1} \colon t \in \mathbb{Z}\},$$
$$k_2(1)_*\{v_1^2 1_{4t+2} \colon t \in \mathbb{Z}\} \quad \text{and} \quad k_4(1)_*\{1_{4t} \colon t \in \mathbb{Z}\}.$$
- $E(2)_2^{1,*}(V_4)$ is a direct sum of
$$k_1(1)_*\{(h_1)_{2t}, v_1^3(h_1)_{2t+1}, (\rho_2)_{2t+1} \colon t \in \mathbb{Z}\},$$
$$k_2(1)_*\{(h_0)_{4t+1}, (h_0)_{4t+2}, (h_0)_{4t+3}, v_1^2(\rho_2)_{4t+2} \colon t \in \mathbb{Z}\},$$
$$k_3(1)_*\{v_1(\zeta_1)_{4t+3} \colon t \in \mathbb{Z}\} \quad \text{and}$$
$$k_4(1)_*\{(h_0)_{4t}, (\zeta_1)_{2t}, (\rho_2)_{4t}, (\zeta_1)_{4t+1} \colon t \in \mathbb{Z}\}.$$
- $E(2)_2^{2,*}(V_4)$ is a direct sum of
$$k_1(1)_*\{(h_1^2)_{2t}, (\rho_2 h_1)_{2t}, v_1^3(h_1^2)_{2t+1}, v_1^3(\rho_2 h_1)_{2t+1} \colon t \in \mathbb{Z}\},$$
$$k_2(1)_*\{(h_0^2)_{4t+1}, (h_0^2)_{4t+2}, (\rho_2 h_0)_{4t+1}, v_1^2(h_0^2)_{4t+3}, v_1^2(\rho_2 h_0)_{4t+2}, v_1^2(\rho_2 h_0)_{4t+3} \colon t \in \mathbb{Z}\},$$
$$k_3(1)_*\{(\zeta_1^2)_{4t+2}, v_1(\zeta_2)_{4t+3}, v_1(\rho_2\zeta_1)_{4t+3} \colon t \in \mathbb{Z}\} \quad \text{and,}$$
$$k_4(1)_*\{(h_0^2)_{2t}, (\zeta_2)_{4t+1}, (\zeta_2)_{2t}, (\zeta_1^2)_{2t+1}, (\zeta_1^2)_{4t}, (\rho_2 h_0)_{4t}, (\rho_2\zeta_1)_{4t+1}, (\rho_2\zeta_1)_{2t} \colon t \in \mathbb{Z}\}.$$
- $E(2)_2^{3,*}(V_4)$ is a direct sum of
$$k_1(1)_*\{(h_0^3)_{2t+1}, (\rho_2 h_1^2)_{2t}, v_1^3(\beta)_{2t+1}, v_1^3(\rho_2 h_1^2)_{2t+1} \colon t \in \mathbb{Z}\},$$
$$k_2(1)_*\{(h_0^3)_{4t+2}, (\rho_2 h_0^2)_{4t+1}, (\rho_2 h_0^2)_{4t+2}, v_1^2(\beta)_{4t+2}, v_1^2(\rho_2 h_0^2)_{4t+3} \colon t \in \mathbb{Z}\},$$
$$k_3(1)_*\{(\zeta_1\zeta_2)_{4t+2}, (\rho_2\zeta_1^2)_{4t+2}, v_1(\zeta_3)_{4t+3}, v_1(\rho_2\zeta_2)_{4t+3} \colon t \in \mathbb{Z}\},$$
$$k_4(1)_*\{(h_0^3)_{4t}, (\zeta_1\zeta_2)_{8t+1}, (\zeta_1\zeta_2)_{4t}, (\zeta_3)_{4t+1}, (\zeta_3)_{2t}, (\beta)_{4t}, (\rho_2 h_0^2)_{4t}, (\rho_2\zeta_2)_{2t}, (\rho_2\zeta_1^2)_{2t+1}, (\rho_2\zeta_1^2)_{4t}, (\rho_2\zeta_2)_{4t+1} \colon t \in \mathbb{Z}\},$$
$$k_4(1)_*\{(\zeta_1\zeta_2)_{2t+1} \colon t/2^{\nu(t)} \equiv 3 \mod (4)\} \quad \text{and}$$
$$k_4(1)_*\{(\zeta_3)_0, (\rho_2\zeta_2)_0, (\rho_2\zeta_1^2)_1\}.$$

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2. $H^s M_1^1$ for $s \le 3$

In this section, we review the structure of $H^* M_1^1$ determined by Shimomura [1].

Theorem 2 ([3] (*cf.* [1])) As a $K(2)_*$-module,

$$H^* K(2)_* = P(g) \otimes E(\rho_2) \otimes K^*,$$

where $P(-)$ and $E(-)$ are polynomial and exterior algebras, respectively. Here,

$$K^* = M \otimes E(\beta) \oplus N \otimes E(\zeta_1)$$

for

$$\begin{aligned} M &= K(2)_*[h_0, h_1]/(h_0 h_1, v_2 h_0^3 - h_1^3), \\ N &= K(2)_*\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\} \end{aligned}$$

with

$$|h_0| = (1,2),\ |h_1| = (1,4),\ |\rho_2| = (1,0),\ |\beta| = (3,0),\ |g| = (4,0),$$
$$|\zeta_i| = (i,0) \text{ for } 1 \le i \le 4.$$

The homomorphism $(-/v_1^k)\colon M_2^0 \to M_1^1; x \mapsto x/v_1^k$ induces $(-/v_1^k)\colon H^* M_2^0 \to H^* M_1^1$. Note that, for $x \in H^* M_2^0$, we have $v_1^k (x/v_1^k) = 0$. Put

$$\begin{aligned} k(1)_* &= BP_*/(2, v_2, v_3, \dots) = \mathbb{Z}/2[v_1], \\ K(1)_* &= E(1)_*/(2) = \mathbb{Z}/2[v_1^{\pm 1}]. \end{aligned}$$

For $x \in H^* M_2^0$ and $n > 0$, Shiomumura defined the subalgebra

$$P(n) = k(1)_*[v_2^{\pm 2^n}] = \mathbb{Z}/2[v_1, v_2^{\pm 2^n}]$$

of $E(2)_*$. He also denote by $Q(A)$ for a subset A of $H^* M_2^0$ the direct sum of $K(1)_*/k(1)_*$ generated by x/v_1^j for $j > 0$ with $x \in A$. Consider the sets

$$\begin{aligned} F_1 &= \mathbb{Z}/2[h_1]/(h_1^3) \otimes E(\beta), \\ F_2 &= E(h_0, v_2 h_0, \beta), \\ F_3 &= \{v_1^{3-i} h_0^i\colon 0 \le i \le 3\}, \\ F(0) &= \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}, \\ F(n) &= v_2^{2^{n-1}} F(0) \text{ for } n > 0, \\ F(n)^* &= v_2^{2^{n-1}+1}\{v_1^i \zeta_1 \zeta_{1+i}\colon 0 \le i \le 3\}. \end{aligned}$$

Theorem 3 ([1, Th. 2.6]) As a $k(1)_*$-module,

$$H^* M_1^1 = P(g) \otimes E(\rho_2) \otimes B^*.$$

Here, B^* is the direct sum of

$$F_1 \otimes P(1)\{v_2/v_1\},\ F_2 \otimes P(2)\{v_2^2/v_1^2\},\ F(1) \otimes P(2)\{v_2^2/v_1^3\},$$
$$(F(1) \oplus F(n)) \otimes P(n+1)\{v_2^{2^n}/v_1^{3\cdot 2^{n-1}}\} \text{ for } n > 1,$$
$$(F_3 \oplus F(n)^*) \otimes P(n+1)\{v_2^{2^n}/v_1^{3\cdot 2^{n-1}+3}\} \text{ for } n > 1,$$

and $Q(I)$, where

$$I = \Big(\mathbb{Z}/2[h_0]/(h_0^4) \otimes E(\beta) \oplus (F(0) \oplus F(1)) \otimes E(\zeta_1)\Big) - \{0\}.$$

Consider the notation of Behrens' type: For an element $x \in H^* M_2^0$, the element $x_{s/t} \in H^* M_1^1$ is defined by

$$v_1^{t-1} x_{s/t} = v_2^s x / v_1.$$

From the above theorem, we obtain that $H^0 M_1^1$ is the direct sum of

$$\begin{aligned} &P(1)\{v_2/v_1\} = k(1)_*\{1_{2t+1/1}\colon t \in \mathbb{Z}\}, \\ &P(2)\{v_2^2/v_1^2\} = k(1)_*\{1_{4t+2/2}\colon t \in \mathbb{Z}\}, \\ &v_1^3 P(n+1)\{v_2^{2^n}/v_1^{3\cdot 2^{n-1}+3}\} \\ &\quad = k(1)_*\{1_{2^{n+1}t+2^n/3\cdot 2^{n-1}}\colon t \in \mathbb{Z}\} \text{ for } n > 1, \end{aligned}$$

and $Q\{1\}$. For a nonzero integer t, we denote

$$\nu(t) = \max\{i \in \mathbb{Z}\colon 2^i \mid t\}.$$

Then, we have the following:

Proposition 4 As a $k(1)_*$-module,

$$H^0 M_1^1 = k(1)_*\{1_{t/a_0(t)}\colon t \ne 0\} \oplus K(1)_*/k(1)_*.$$

Here, the summand $K(1)_*/k(1)_*$ is generated by $1_{0/j}$ for $j > 0$, and

$$a_0(t) = \begin{cases} 1 & t \equiv 1, 3 \mod (4), \\ 2 & t \equiv 2 \mod (4), \\ 3 \cdot 2^{\nu(t)-1} & t \equiv 0 \mod (4). \end{cases}$$

Next turn to $H^1 M_1^1$. This module is the direct sum of

$$\begin{aligned} &h_1 P(1)\{v_2/v_1\} = k(1)_*\{(h_1)_{2t+1/1}\colon t \in \mathbb{Z}\}, \\ &\{h_0, v_2 h_0\} \otimes P(2)\{v_2^2/v_1^2\} = k(1)_*\{(h_0)_{4t+2/2},\ (h_0)_{4t+3/2}\colon t \in \mathbb{Z}\}, \\ &v_2 \zeta_1 P(2)\{v_2^2/v_1^3\} = k(1)_*\{(\zeta_1)_{4t+3/3}\colon t \in \mathbb{Z}\}, \\ &\{v_2\zeta_1, v_2^{2^{n-1}}\zeta_1\} \otimes P(n+1)\{v_2^{2^n}/v_1^{3\cdot 2^{n-1}}\} \\ &\quad = k(1)_*\{(\zeta_1)_{2^n(2t+1)+1/3\cdot 2^{n-1}}, (\zeta_1)_{2^{n-1}(4t+3)/3\cdot 2^{n-1}}\colon t \in \mathbb{Z}\} \text{ for } n > 1, \\ &v_1^2 h_0 P(n+1)\{v_2^{2^n}/v_1^{3\cdot 2^{n-1}+3}\} = \{(h_0)_{2^n(2t+1)/3\cdot 2^{n-1}+1}\} \text{ for } n > 1, \end{aligned}$$

$Q\{h_0, \zeta_1, v_2\zeta_1\}$ and $\rho_2 (B^*)^0 = \rho_2 H^0 M_1^1$. Therefore, we have the following:

Proposition 5 As a $k(1)_*$-module, $H^1 M_1^1$ is the direct sum of

$$\begin{aligned} &k(1)_*\{(h_0)_{t/a_1(t)}\colon 0 \ne t \equiv 0, 2, 3 \mod (4)\}, \\ &k(1)_*\{(h_1)_{2t+1/1}\colon t \in \mathbb{Z}\}, \\ &k(1)_*\{(\zeta_1)_{t/a_1'(t)}\colon 1 \ne t \equiv 1 \mod (4), \text{ or } t/2^{\nu(t)} \equiv 3 \mod (4)\}, \\ &k(1)_*\{(\rho_2)_{t/a_0(t)}\colon t \ne 0\} \end{aligned}$$

and four copies of $K(1)_*/k(1)_*$ generated by $(h_0)_{0/j}$, $(\zeta_1)_{0/j}$, $(\zeta_1)_{1/j}$ and $(\rho_2)_{0/j}$ for $j > 0$. Here,

$$a_1(t) = \begin{cases} 2 & t \equiv 2, 3 \mod (4), \\ 3 \cdot 2^{\nu(t)-1} + 1 & 0 \ne t \equiv 0 \mod (4), \end{cases}$$

$$a_1'(t) = \begin{cases} 3 \cdot 2^{\nu(t-1)-1} & 1 \ne t \equiv 1 \mod (4), \\ 3 \cdot 2^{\nu(t)} & t/2^{\nu(t)} \equiv 3 \mod (4). \end{cases}$$

Next we consider $H^2 M_1^1$. This is the directs sum of

$$\begin{aligned} &h_1^2 P(1)\{v_2/v_1\} = k(1)_*\{(h_1^2)_{2t+1/1}\colon t \in \mathbb{Z}\}, \\ &v_2 h_0^2 P(2)\{v_2^2/v_1^2\} = k(1)_*\{(h_0^2)_{4t+3/2}\colon t \in \mathbb{Z}\}, \\ &v_2 \zeta_2 P(2)\{v_2^2/v_1^3\} = k(1)_*\{(\zeta_2)_{4t+3/3}\colon t \in \mathbb{Z}\}, \\ &\{v_2\zeta_2, v_2^{2^{n-1}}\zeta_2\} \otimes P(n+1)\{v_2^{2^n}/v_1^{3\cdot 2^{n-1}}\} \\ &\quad = k(1)_*\{(\zeta_2)_{2^n(2t+1)+1/3\cdot 2^{n-1}}, (\zeta_2)_{2^{n-1}(4t+3)/3\cdot 2^{n-1}}\colon t \in \mathbb{Z}\} \text{ for } n > 1, \\ &\{v_1 h_0^2, v_2^{2^{n-1}+1}\zeta_1^2\} \otimes P(n+1)\{v_2^{2^n}/v_1^{3\cdot 2^{n-1}+3}\} \\ &\quad = k(1)_*\{(h_0^2)_{2^n(2t+1)/3\cdot 2^{n-1}+2}, (\zeta_1^2)_{2^{n-1}(4t+3)+1/3\cdot 2^{n-1}+3}\colon t \in \mathbb{Z}\} \text{ for } n > 1, \end{aligned}$$

$Q\{h_0^2, \zeta_2, v_2\zeta_2, \zeta_1^2, v_2\zeta_1^2\}$ and $\rho_2(B^*)^1$. Therefore, we have the following:

Proposition 6 As a $k(1)_*$-module, $H^2M_1^1$ is the direct sum of

$$\begin{aligned}
&k(1)_*\{(h_0^2)_{t/a_2(t)} \colon 0 \neq t \equiv 0, 3 \mod (4)\},\\
&k(1)_*\{(h_1^2)_{2t+1/1} \colon t \in \mathbb{Z}\},\\
&k(1)_*\{(\zeta_2)_{t/a_1'(t)} \colon 1 \neq t \equiv 1 \mod (4), \text{ or } t/2^{\nu(t)} \equiv 3 \mod (4)\},\\
&k(1)_*\{(\zeta_1^2)_{t/a_2'(t)} \colon\\
&\quad 1 \neq t \equiv 1 \mod (2) \text{ and } (t-1)/2^{\nu(t-1)} \equiv 3 \mod (4)\},\\
&k(1)_*\{(\rho_2 h_0)_{t/a_1(t)} \colon 0 \neq t \equiv 0, 2, 3 \mod (4)\},\\
&k(1)_*\{(\rho_2 h_1)_{2t+1/1} \colon t \in \mathbb{Z}\},\\
&k(1)_*\{(\rho_2\zeta_1)_{t/a_1'(t)} \colon\\
&\quad 1 \neq t \equiv 1 \mod (4), \text{ or } t/2^{\nu(t)} \equiv 3 \mod (4)\}
\end{aligned}$$

and eight copies of $K(1)_*/k(1)_*$ generated by $(h_0^2)_{0/j}$, $(\zeta_2)_{0/j}$, $(\zeta_2)_{1/j}$, $(\zeta_1^2)_{0/j}$, $(\zeta_1^2)_{1/j}$, $(\rho_2 h_0)_{0/j}$, $(\rho_2\zeta_1)_{0/j}$ and $(\rho_2\zeta_1)_{1/j}$ for $j > 0$. Here,

$$\begin{aligned}
a_2(t) &= \begin{cases} 2 & t \equiv 3 \mod (4), \\ 3 \cdot 2^{\nu(t)-1} + 2 & 0 \neq t \equiv 0 \mod (4), \end{cases}\\
a_2'(t) &= 3 \cdot 2^{\nu(t-1)} + 3.
\end{aligned}$$

The third cohomology $H^3M_1^1$ is the direct sum of

$$\begin{aligned}
&\beta P(1)\{v_2/v_1\} = k(1)_*\{(\beta)_{2t+1/1} \colon t \in \mathbb{Z}\},\\
&\beta P(2)\{v_2^2/v_1^2\} = k(1)_*\{(\beta)_{4t+2/2} \colon t \in \mathbb{Z}\},\\
&v_2\zeta_3 P(2)\{v_2^2/v_1^3\} = k(1)_*\{(\zeta_3)_{4t+2/2} \colon t \in \mathbb{Z}\},\\
&\{v_2\zeta_3, v_2^{2^{n-1}}\zeta_3\} \otimes P(n+1)\{v_2^{2^n}/v_1^{3\cdot 2^{n-1}}\}\\
&\quad = k(1)_*\{(\zeta_3)_{2^n(2t+1)+1/3\cdot 2^{n-1}}, (\zeta_3)_{2^{n-1}(4t+3)/3\cdot 2^{n-1}}\} \text{ for } n > 1,\\
&\{h_0^3, v_1 v_2^{2^{n-1}+1}\zeta_1\zeta_2\} \otimes P(n+1)\{v_2^{2^n}/v_1^{3\cdot 2^{n-1}+3}\}\\
&= k(1)_*\{(h_0^3)_{2^n(2t+1)/3\cdot 2^{n-1}+3}, (\zeta_1\zeta_2)_{2^{n-1}(4t+3)+1/3\cdot 2^{n-1}+2}\} \text{ for } n > 1,
\end{aligned}$$

$Q\{h_0^3, \beta, \zeta_3, v_2\zeta_3, \zeta_1\zeta_2, v_2\zeta_1\zeta_2\}$ and $\rho_2(B^*)^2$.

Proposition 7 As a $k(1)_*$-module, $H^3M_1^1$ is the direct sum of

$$\begin{aligned}
&k(1)_*\{(\beta)_{t/a_3(t)} \colon t \equiv 1, 2, 3 \mod (4)\},\\
&k(1)_*\{(\zeta_3)_{t/a_3'(t)} \colon t \equiv 1 \mod (4), \text{ or } t/2^{\nu(t)} \equiv 3 \mod (4)\},\\
&k(1)_*\{(h_0^3)_{t/a_3'(t)} \colon 0 \neq t \equiv 0 \mod (4)\},\\
&k(1)_*\{(\zeta_1\zeta_2)_{t/a_3''(t)} \colon\\
&\quad 1 \neq t \equiv 1 \mod (2) \text{ and } (t-1)/2^{\nu(t-1)} \equiv 3 \mod (4)\},\\
&k(1)_*\{(\rho_2 h_0^2)_{t/a_2(t)} \colon 0 \neq t \equiv 0, 3 \mod (4)\},\\
&k(1)_*\{(\rho_2 h_1^2)_{2t+1/1} \colon t \in \mathbb{Z}\},\\
&k(1)_*\{(\rho_2\zeta_2)_{t/a_1'(t)} \colon\\
&\quad 1 \neq t \equiv 1 \mod (4) \text{ or } t/2^{\nu(t)} \equiv 3 \mod (4)\},\\
&k(1)_*\{(\rho_2\zeta_1^2)_{t/a_2'(t)} \colon\\
&\quad 1 \neq t \equiv 1 \mod (2) \text{ and } (t-1)/2^{\nu(t-1)} \equiv 3 \mod (4)\},
\end{aligned}$$

and eleven copies of $K(1)_*/k(1)_*$ generated by $(h_0^3)_{0/j}$, $(\beta)_{0/j}$, $(\zeta_3)_{0/j}$, $(\zeta_3)_{1/j}$, $(\zeta_1\zeta_2)_{0/j}$, $(\zeta_1\zeta_2)_{1/j}$, $(\rho_2 h_0^2)_{0/j}$, $(\rho_2\zeta_2)_{0/j}$, $(\rho_2\zeta_2)_{1/j}$, $(\rho_2\zeta_1^2)_{0/j}$, $(\rho_2\zeta_1^2)_{1/j}$ for $j > 0$. Here,

$$\begin{aligned}
a_3(t) &= \begin{cases} 1 & t \equiv 1, 3 \mod (4), \\ 2 & t \equiv 2 \mod (4), \end{cases}\\
a_3'(t) &= 3 \cdot 2^{\nu(t)-1} + 3,\\
a_3''(t) &= 3 \cdot 2^{\nu(t-1)} + 2.
\end{aligned}$$

3. $E(2)^{s,*}(V_4)$ for $s \leq 3$

We put

$$K_4(2)_* = E(2)_*(V_4) = E(2)_*/(2, v_1^4).$$

The short exact sequences

$$\begin{aligned}
&0 \to M_2^0 \xrightarrow{-/v_1} M_1^1 \xrightarrow{v_1} M_1^1 \to 0 \text{ and}\\
&0 \to K_4(2)_* \xrightarrow{-/v_1^4} M_1^1 \xrightarrow{v_1^4} M_1^1 \to 0,
\end{aligned}$$

induce the connecting homomorphims

$$\delta \colon H^*M_1^1 \to H^{*+1}M_2^0 \quad \text{and} \quad \partial \colon H^*M_1^1 \to H^{*+1}K_4(2)_*$$

respectively.

Lemma 8 Suppose that $x \in H^*M_2^0$. If $\delta(x_{s/t}) = v_2^u y$, then $\partial(x_{s/t}) = (y)_u$.

Proof. Let $\tau \colon K_4(2)_* \to M_2^0$ be the canonical projection. Since the diagram

$$\begin{array}{ccccc}
M_2^0 & \xrightarrow{-/v_1} & M_1^1 & \xrightarrow{v_1} & M_1^1\\
\tau\uparrow & & v_1^3\uparrow & & \|\\
K_4(2)_* & \xrightarrow[-/v_1^4]{} & M_1^1 & \xrightarrow[v_1^4]{} & M_1^1
\end{array}$$

commutes, if $\delta(x_{s/t}) = v_2^u y$, then

$$\rho_*\partial(x_{s/t}) = \delta(x_{s/t}) = v_2^u y.$$

Therefore, $\partial(x_{s/t}) = (y)_u$. □

From [1, Lem. 4.6], we obtain the following:

Lemma 9

$$\delta(1_{t/a_0(t)}) = \begin{cases} v_2^{t-1}h_1 & t \equiv 1,3 \mod (4), \\ v_2^{t-1}h_0 & t \equiv 2 \mod (4), \\ v_2^{t-2^{\nu(t)}-1}\zeta_1 & 0 \neq t \equiv 0 \mod (4), \end{cases}$$

$$\delta((h_0)_{t/a_1(t)}) = \begin{cases} v_2^{t-1}h_0^2 & t \equiv 2,3 \mod (4), \\ v_2^{t-2^{\nu(t)}-1}\zeta_2 & 0 \neq t \equiv 0 \mod (4), \end{cases}$$

$$\delta((h_1)_{2t+1/1}) = v_2^{2t}h_1^2,$$

$$\delta((\zeta_1)_{t/a_1'(t)}) = \begin{cases} v_2^{t-2^{\nu(t-1)}-1}\zeta_1^2 & 1 \neq t \equiv 1 \mod (4), \\ v_2^{t-\nu(t)}\zeta_1^2 & t/2^{\nu(t)} \equiv 3 \mod (4), \end{cases}$$

$$\delta((\rho_2)_{t/a_0(t)}) = \rho_2\delta(1_{t/a_0(t)}) = \begin{cases} v_2^{t-1}\rho_2 h_1 & t \equiv 1,3 \mod (4), \\ v_2^{t-1}\rho_2 h_0 & t \equiv 2 \mod (4), \\ v_2^{t-2^{\nu(t)}-1}\rho_2\zeta_1 & 0 \neq t \equiv 0 \mod (4), \end{cases}$$

$$\delta((h_0^2)_{t/a_2(t)}) = \begin{cases} v_2^{t-1}h_0^3 & t \equiv 3 \mod (4), \\ v_2^{t-2^{\nu(t)}-1}\zeta_3 & 0 \neq t \equiv 0 \mod (4), \end{cases}$$

$$\delta((h_1^2)_{2t+1/1}) = v_2^{t}h_0^3,$$

$$\delta((\zeta_2)_{t/a_1'(t)}) = \begin{cases} v_2^{t-2^{\nu(t-1)}}\zeta_1\zeta_2 & 1 \neq t \equiv 1 \mod (4), \\ v_2^{t-2^{\nu(t)}}\zeta_1\zeta_2 & t/2^{\nu(t)} \equiv 3 \mod (4), \end{cases}$$

$$\delta((\zeta_1^2)_{t/a_2'(t)}) = v_2^{t-1-2^{\nu(t-1)}}\beta,$$

$$\delta((\rho_2 h_0)_{t/a_1(t)}) = \rho_2\delta((h_0)_{t/a_1(t)}) = \begin{cases} v_2^{t-1}\rho_2 h_0^2 & t \equiv 2,3 \mod (4), \\ v_2^{t-2^{\nu(t)}-1}\rho_2\zeta_2 & 0 \neq t \equiv 0 \mod (4), \end{cases}$$

$$\delta((\rho_2 h_1)_{2t+1/1}) = v_2^{2t}\rho_2 h_1^2,$$

$$\delta((\rho_2\zeta_1)_{t/a_1'(t)}) = \begin{cases} v_2^{t-2^{\nu(t-1)}-1}\rho_2\zeta_1^2 & 1 \neq t \equiv 1 \mod (4), \\ v_2^{t-\nu(t)}\rho_2\zeta_1^2 & t/2^{\nu(t)} \equiv 3 \mod (4). \end{cases}$$

Proof of Theorem 1. Consider the exact sequence $0 \to H^0K_4(2)_* \xrightarrow{-/v_1^4} H^0M_1^1 \xrightarrow{v_1^4} H^0M_1^1$. This implies

$$H^0K_4(2)_* \xrightarrow[\sim]{-/v_1^4} \ker(v_1^4).$$

By Proposition 4,

$$H^0K_4(2)_* \xrightarrow[\sim]{-/v_1^4} k(1)_*\{1_{t/\min\{4,a_0(t)\}}, 1_{0/4} \colon t \neq 0\},$$

and we see $H^0K_4(2)_*$ in the statement.

Next turn to $H^1K_4(2)_*$. We consider he long exact sequence

$$\cdots \xrightarrow{v_1^4} H^0M_1^1 \xrightarrow{\partial} H^1K_4(2)_* \xrightarrow{-/v_1^4} H^1M_1^1 \xrightarrow{v_1^4} \cdots.$$

Put $k_\ell(1)_* = \mathbb{Z}/2[v_1]/(v_1^\ell)$ for $\ell \geq 1$. By Proposition 4, we have $(H^0M_1^1)/(v_1^4) = k_{\min\{4,a_0(t)\}}(1)_*\{1_{t/a_0(t)} \colon t \neq 0\}$. By Lemma 8 and Lemma 9,

$$\partial(1_{t/a_0(t)}) = \begin{cases} (h_1)_{t-1} & t \equiv 1,3 \mod (4), \\ (h_0)_{t-1} & t \equiv 2 \mod (4), \\ (\zeta_1)_{t-2^{\nu(t)-1}} & 0 \neq t \equiv 0 \mod (4). \end{cases}$$

Therefore, $\partial\left((H^0M_1^1)/(v_1^4)\right)$ is a direct sum of

$k_1(1)_*\{(h_1)_{2t} \colon t \in \mathbb{Z}\}$,
$k_2(1)_*\{(h_0)_{4t+1} \colon t \in \mathbb{Z}\}$,
$k_4(1)_*\{(\zeta_1)_{2t} \colon t/2^{\nu(t)} \equiv 1 \mod (4)\}$.

By Proposition 5, the kernel of $v_1^4 \colon H^1M_1^1 \to H^1M_1^1$ is a direct sum of

$k(1)_*\{(h_0)_{t/2} \colon t \equiv 2,3 \mod (4)\}$,
$k(1)_*\{(h_0)_{4t/4} \colon t \neq 0\}$,
$k(1)_*\{(h_1)_{2t+1/1} \colon t \in \mathbb{Z}\}$,
$k(1)_*\{(\zeta_1)_{4t+3/3} \colon t \in \mathbb{Z}\}$,
$k(1)_*\{(\zeta_1)_{t/4} \colon 1 \neq t \equiv 1 \mod (4)\}$,
$k(1)_*\{(\zeta_1)_{2t/4} \colon t/2^{\nu(t)} \equiv 3 \mod (4)\}$,
$k(1)_*\{(\rho_2)_{2t+1/1} \colon t \in \mathbb{Z}\}$,
$k(1)_*\{(\rho_2)_{4t+2/2} \colon t \in \mathbb{Z}\}$,
$k(1)_*\{(\rho_2)_{4t/4} \colon t \neq 0\}$

and $k(1)_*\{(h_0)_{0/4}, (\zeta_1)_{0/4}, (\zeta_1)_{1/4}, (\rho_2)_{0/4}\}$. From the short exact sequence

$$0 \to (H^0M_1^1)/(v_1^4) \xrightarrow{\partial} H^1K_4(2)_* \xrightarrow{-/v_1^4} \ker(v_1^4) \to 0,$$

we obtain $H^1K_4(2)_*$.

For $H^2K_4(2)_*$, we consider the short exact sequence

$$0 \to (H^1M_1^1)/(v_1^4) \xrightarrow{\partial} H^2K_4(2)_* \xrightarrow{-/v_1^4} \ker(v_1^4) \to 0.$$

Lemma 8 and Lemma 9 imply that

$$\partial((h_0)_{t/a_1(t)}) = \begin{cases} (h_0^2)_{t-1} & t \equiv 2,3 \mod (4), \\ (\zeta_2)_{t-2^{\nu(t)-1}} & 0 \neq t \equiv 0 \mod (4), \end{cases}$$

$$\partial((h_1)_{2t+1/1}) = (h_1^2)_{2t},$$

$$\partial((\zeta_1)_{t/a_1'(t)}) = \begin{cases} (\zeta_1^2)_{t-2^{\nu(t-1)-1}} & 1 \neq t \equiv 1 \mod (4), \\ (\zeta_1^2)_{t-2^{\nu(t)}} & t/2^{\nu(t)} \equiv 3 \mod (4), \end{cases}$$

$$\partial((\rho_2)_{t/a_0(t)}) = \begin{cases} (\rho_2 h_1)_{t-1} & t \equiv 1,3 \mod (4), \\ (\rho_2 h_0)_{t-1} & t \equiv 2 \mod (4), \\ (\rho_2\zeta_1)_{t-2^{\nu(t)-1}} & 0 \neq t \equiv 0 \mod (4). \end{cases}$$

Then, $\partial\left((H^1M_1^1)/(v_1^4)\right)$ is a direct sum of

$k_2(1)_*\{(h_0^2)_t \colon t \equiv 1,2 \mod (4)\}$,
$k_4(1)_*\{(\zeta_2)_{2t}, (\zeta_1^2)_{2t+1}, (\rho_2\zeta_1)_{2t} \colon t/2^{\nu(t)} \equiv 1 \mod (4)\}$,
$k_1(1)_*\{(h_1^2)_{2t}, (\rho_2 h_1)_{2t} \colon t \in \mathbb{Z}\}$,
$k_3(1)_*\{(\zeta_1^2)_{4t+2} \colon t \in \mathbb{Z}\}$,
$k_4(1)_*\{(\zeta_1^2)_{4t} \colon t \neq 0\}$.
$k_2(1)_*\{(\rho_2 h_0)_{4t+1} \colon t \in \mathbb{Z}\}$.

Proposition 6 implies that the kernel of $v_1^4 \colon H^2M_1^1 \to H^2M_1^1$ is a direct sum of

$k(1)_*\{(h_0^2)_{4t+3/2} \colon t \in \mathbb{Z}\}$,
$k(1)_*\{(h_0^2)_{4t/4} \colon t \neq 0\}$,
$k(1)_*\{(h_1^2)_{2t+1/1} \colon t \in \mathbb{Z}\}$,
$k(1)_*\{(\zeta_2)_{4t+3/3} \colon t \in \mathbb{Z}\}$,
$k(1)_*\{(\zeta_2)_{4t+1/4} \colon t \neq 0\}$,
$k(1)_*\{(\zeta_2)_{2t/4} \colon t/2^{\nu(t)} \equiv 3 \mod (4)\}$,
$k(1)_*\{(\zeta_1^2)_{2t+1/4} \colon t/2^{\nu(t)} \equiv 3 \mod (4)\}$,
$k(1)_*\{(\rho_2 h_0)_{t/2} \colon t \equiv 2,3 \mod (4)\}$,
$k(1)_*\{(\rho_2 h_0)_{4t/4} \colon t \neq 0\}$,
$k(1)_*\{(\rho_2 h_1)_{2t+1/1} \colon t \in \mathbb{Z}\}$,
$k(1)_*\{(\rho_2\zeta_1)_{4t+3/3} \colon t \in \mathbb{Z}\}$,
$k(1)_*\{(\rho_2\zeta_1)_{4t+1/4} \colon t \neq 0\}$,
$k(1)_*\{(\rho_2\zeta_1)_{2t/4} \colon t/2^{\nu(t)} \equiv 3 \mod (4)\}$ and
$k(1)_*\{(h_0^2)_{0/4}, (\zeta_2)_{0/4}, (\zeta_2)_{1/4}, (\zeta_1^2)_{0/4}, (\zeta_1^2)_{1/4},$
$(\rho_2 h_0)_{0/4}, (\rho_2\zeta_1)_{0/4}, (\rho_2\zeta_1)_{1/4}\}$.

Then, we see the structure of $H^2K_4(2)_*$.

Finally, we consider $H^3K_4(2)_*$. We consider the short exact sequence

$$0 \to (H^2M_1^1)/(v_1^4) \xrightarrow{\partial} H^3K_4(2)_* \xrightarrow{-/v_1^4} \ker(v_1^4) \to 0.$$

By Lemma 8 and Lemma 9, we have

$$\partial((h_0^2)_{t/a_2(t)}) = \begin{cases} (h_0^3)_{t-1} & t \equiv 3 \mod (4), \\ (\zeta_3)_{t-2^{\nu(t)}-1} & 0 \neq t \equiv 0 \mod (4), \end{cases}$$
$$\partial((h_1^2)_{2t+1/1}) = (h_0^3)_{2t+1},$$
$$\partial((\zeta_2)_{t/a_1'(t)}) = \begin{cases} (\zeta_1\zeta_2)_{t-2^{\nu(t-1)}} & 1 \neq t \equiv 1 \mod (4), \\ (\zeta_1\zeta_2)_{t-2^{\nu(t)}} & t/2^{\nu(t)} \equiv 3 \mod (4), \end{cases}$$
$$\partial((\zeta_1^2)_{t/a_2'(t)}) = (\beta)_{t-1-2^{\nu(t-1)}},$$
$$\partial((\rho_2h_0)_{t/a_1(t)}) = \begin{cases} (\rho_2h_0^2)_{t-1} & t \equiv 2,3 \mod (4), \\ (\rho_2\zeta_2)_{t-2^{\nu(t)}-1} & 0 \neq t \equiv 0 \mod (4), \end{cases}$$
$$\partial((\rho_2h_1)_{2t+1/1}) = (\rho_2h_1^2)_{2t},$$
$$\partial((\rho_2\zeta_1)_{t/a_1'(t)}) = \begin{cases} (\rho_2\zeta_1^2)_{t-2^{\nu(t-1)}-1} & 1 \neq t \equiv 1 \mod (4), \\ (\rho_2\zeta_1^2)_{t-2^{\nu(t)}} & t/2^{\nu(t)} \equiv 3 \mod (4). \end{cases}$$

Then, $\partial((H^2M_1^1)/(v_1^4))$ is a direct sum of

$k_2(1)_*\{(h_0^3)_{4t+2} : t \in \mathbb{Z}\}$,
$k_4(1)_*\{(\zeta_3)_{2t} : t/2^{\nu(t)} \equiv 1 \mod (4)\}$,
$k_1(1)_*\{(h_0^3)_{2t+1} : t \in \mathbb{Z}\}$,
$k_3(1)_*\{(\zeta_1\zeta_2)_{4t+2} : t \in \mathbb{Z}\}$,
$k_4(1)_*\{(\zeta_1\zeta_2)_{8t+1} : t \neq 0\}$,
$k_4(1)_*\{(\zeta_1\zeta_2)_{4t} : t \neq 0\}$,
$k_4(1)_*\{(\beta)_{4t} : t \neq 0\}$,
$k_2(1)_*\{(\rho_2h_0^2)_t : t \equiv 1,2 \mod (4)\}$,
$k_4(1)_*\{(\rho_2\zeta_2)_{2t} : t/2^{\nu(t)} \equiv 1 \mod (4)\}$,
$k_1(1)_*\{(\rho_2h_1^2)_{2t} : t \in \mathbb{Z}\}$,
$k_3(1)_*\{(\rho_2\zeta_1^2)_{4t+2} : t \in \mathbb{Z}\}$,
$k_4(1)_*\{(\rho_2\zeta_1^2)_{2t+1} : t/2^{\nu(t)} \equiv 1 \mod (4)\}$ and
$k_4(1)_*\{(\rho_2\zeta_1^2)_{4t} : t \neq 0\}$.

The kernel of $v_1^4 : H^3M_1^1 \to H^3M_1^1$ is a direct sum of

$k(1)_*\{(\beta)_{2t+1/1}, (\beta)_{4t+2/2} : t \in \mathbb{Z}\}$,
$k(1)_*\{(\zeta_3)_{4t+3/3} : t \in \mathbb{Z}\}$,
$k(1)_*\{(\zeta_3)_{4t+1/4} : t \in \mathbb{Z}\}$,
$k(1)_*\{(\zeta_3)_{2t/4} : t/2^{\nu(t)} \equiv 3\}$,
$k(1)_*\{(h_0^3)_{4t/4} : t \neq 0\}$,
$k(1)_*\{(\zeta_1\zeta_2)_{2t+1/4} : t/2^{\nu(t)} \equiv 3 \mod (4)\}$,
$k(1)_*\{(\rho_2h_0^2)_{4t+3/2} : t \in \mathbb{Z}\}$,
$k(1)_*\{(\rho_2h_0^2)_{4t/4} : t \neq 0\}$,
$k(1)_*\{(\rho_2h_1^2)_{2t+1/1} : t \in \mathbb{Z}\}$,
$k(1)_*\{(\rho_2\zeta_2)_{4t+3/3} : t \in \mathbb{Z}\}$,
$k(1)_*\{(\rho_2\zeta_2)_{4t+1/4} : t \neq 0\}$,
$k(1)_*\{(\rho_2\zeta_2)_{2t/4} : t/2^{\nu(t)} \equiv 3 \mod (4)\}$,
$k(1)_*\{(\rho_2\zeta_1^2)_{2t+1/4} : t/2^{\nu(t)} \equiv 3 \mod (4)\}$ and
$k(1)_*\{(h_0^3)_{0/4}, (\beta)_{0/4}, (\zeta_3)_{0/4}, (\zeta_1\zeta_2)_{0/4}, (\zeta_1\zeta_2)_{1/4}, (\rho_2h_0^2)_{0/4}, (\rho_2\zeta_2)_{0/4}, (\rho_2\zeta_2)_{1/4}, (\rho_2\zeta_1^2)_{0/4}, (\rho_2\zeta_1^2)_{1/4}\}$.

Therefore, we obtain the structure of $H^3K_4(2)_*$. □

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