A note on the Adams v_1-periodic map at the prime two

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The mod 2 Moore spectrum $M(2)$ admits a v_1-periodic map $v_1^4\colon \Sigma^4 M(2) \to M(2)$, which is called the Adams v_1-periodic map. Let i be the inclusion from the sphere spectrum to the bottom cell of $M(2)$, and j the collapsing map from $M(2)$ to the sphere spectrum. The composite $jv_1^4 i$ is well known as the element α_4 in the stable homotopy groups of spheres, which is explained as Toda's 8σ . In this note, we explain the map v_1^4 by using Toda's notation in [1].

1. Introduction

Let p be a prime number and $K(n)$ the n-th Morava K-theory spectrum at p, that is,

$$K(n)_* = \pi_*(K(n)) = \begin{cases} \pi_0(K(0)) = \mathbb{Q} & n = 0, \\ \mathbb{Z}/p[v_n^{\pm 1}] & n > 0, \end{cases}$$

where $|v_n| = 2(p^n - 1)$. A p-local finite spectrum V is *of type n* if the $K(i)_*$-homology $K(i)_*(V)$ is zero for $i < n$, and $K(n)_*(V) \neq 0$. Periodicity theorem (*cf.* [3, Th. 1.5.4]) claims that, for any finite spectrum V of type n, there exists a self map $v\colon \Sigma^{|v|} V \to V$ such that $K(i)_*(v) = 0$ for $i < n$, and $K(n)_*(v)$ is an isomorphism. We call such map a *v_n-self map* of V.

The *mod k Moore spectrum $M(k)$* for an integer k is defined by the cofiber sequence

$$S^0 \xrightarrow{k} S^0 \xrightarrow{i_k} M(k) \xrightarrow{j_k} S^1.$$

Note that, if s divides t, then we have the following commutative diagram:

$$\begin{array}{ccccccc} S^0 & \xrightarrow{s} & S^0 & \xrightarrow{i_s} & M(s) & \xrightarrow{j_s} & S^1 \\ \| & & \downarrow{\scriptstyle t/s} & & \downarrow{\scriptstyle \lambda_{s,t}} & & \| \\ S^0 & \xrightarrow{t} & S^0 & \xrightarrow{i_t} & M(t) & \xrightarrow{j_t} & S^1 \\ \downarrow{\scriptstyle t/s} & & \| & & \downarrow{\scriptstyle \rho_{t,s}} & & \downarrow{\scriptstyle t/s} \\ S^0 & \xrightarrow{s} & S^0 & \xrightarrow{i_s} & M(s) & \xrightarrow{j_s} & S^1 \end{array}$$

This gives rise to

(1) $$\begin{gathered} \lambda_{s,t} i_s = (t/s) i_t, \quad j_t \lambda_{s,t} = j_s, \\ \rho_{t,s} i_t = i_s \quad \text{and} \quad j_s \rho_{t,s} = (t/s) j_t. \end{gathered}$$

We recall that the mod p Moore spectrum $M(p)$ is a finite spectrum of type 1, and $K(1)_*(M(p)) = K(1)_* = \mathbb{Z}/p[v_1^{\pm 1}]$ with $|v_1| = 2p - 2$. Adams showed that the spectrum $M(p)$ admits a v_1-periodic map

$$v_1^d\colon \Sigma^{2(p-1)d} M(p) \to M(p) \quad \text{for} \quad d = \begin{cases} 4 & p = 2 \\ 1 & p > 2 \end{cases}$$

which induces an isomorphism $K(1)_*(v_1^d) = v_1^d\colon K(1)_* \xrightarrow{\sim} K(1)_*$. In this note, we consider the Adams v_1-periodic map

$$v_1^4\colon \Sigma^8 M(2) \to M(2)$$

at $p = 2$.

Hereafter, we fix $p = 2$. In [1], Toda calculated the homotopy groups $\pi_*(S^0)$ of the sphere spectrum for $* \leq 18$, and he constructed Hopf invariant one elements η, ν and σ. From [1, p.189], we obtain

(2) $$\pi_1(S^0) = \mathbb{Z}/2\{\eta\} \quad \text{and} \quad \pi_7(S^0) = \mathbb{Z}/16\{\sigma\}.$$

Since $16\sigma = 0$, there exists a coextension $\widetilde{\sigma}\colon S^8 \to M(16)$ of σ, that is, $j_{16}\widetilde{\sigma} = \sigma$. Recall that the diagram

(3) $$\begin{array}{ccc} M(2) & \xrightarrow{2} & M(2) \\ {\scriptstyle j_2}\downarrow & & \uparrow{\scriptstyle i_2} \\ S^1 & \xrightarrow{\eta} & S^0 \end{array}$$

commutes. Therefore, by (1) and (2), we have an equation $2\rho_{16,2}\widetilde{\sigma} = i_2\eta j_2 \rho_{16,2}\widetilde{\sigma} = 8 i_2 \eta j_{16}\widetilde{\sigma} = 0$. This implies that the composite $\rho_{16,2}\widetilde{\sigma}\colon S^8 \to M(2)$ has an extension

(4) $$\overline{\rho_{16,2}\widetilde{\sigma}}\colon \Sigma^8 M(2) \to M(2),$$

that is, $\overline{\rho_{16,2}\widetilde{\sigma}}\, i_2 = \rho_{16,2}\widetilde{\sigma}$. In this note, we prove the following result:

Theorem 5 $v_1^4 = \overline{\rho_{16,2}\widetilde{\sigma}}$.

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2. The map $v_1^4\colon \Sigma^8 M(2) \to M(2)$

For (2-localized) spectra X and Y, we denote by $[X, Y]$ the group of morphisms from X to Y. We use the notation 1_X for the identity map of X.

Lemma 6 For $a \geq 1$,

$$[M(2^a), M(2)] = \begin{cases} \mathbb{Z}/4\{1_{M(2)}\} & a = 1, \\ \mathbb{Z}/2\{i_2\eta j_{2^a}\} \oplus \mathbb{Z}/2\{\rho_{2^a,2}\} & a > 1. \end{cases}$$

Proof. Recall that $\pi_{-1}(S^0) = 0$, $\pi_0(S^0) = \mathbb{Z}_{(2)}\{1_{S^0}\}$ and $\pi_1(S^0) = \mathbb{Z}/2\{\eta\}$. From the diagram

$$\begin{array}{ccccc} \pi_1(S^0) & & & & \pi_0(S^0) \\ \downarrow{\scriptstyle (i_2)_*} & & & & \downarrow{\scriptstyle (i_2)_*} \\ \pi_1(M(2)) & \xrightarrow{j_{2^a}^*} & [M(2^a), M(2)] & \xrightarrow{i_{2^a}^*} & \pi_0(M(2)) \\ \downarrow{\scriptstyle (j_2)_*} & & & & \downarrow{\scriptstyle (j_2)_*} \\ \pi_0(S^0) & & & & \pi_{-1}(S^0) \end{array}$$

we obtain $\pi_0(M(2)) = \mathbb{Z}/2\{i_2\}$ and $\pi_1(M(2)) = \mathbb{Z}/2\{i_2\eta\}$. They give rise to the short exact sequence

$$\mathbb{Z}/2\{i_2\eta\} \xrightarrow{j_{2^a}^*} [M(2^a), M(2)] \xrightarrow{i_{2^a}^*} \mathbb{Z}/2\{i_2\}.$$

By (1), the induced homomorphism $i_{2^a}^*$ satisfies that $i_{2^a}^*(\rho_{2^a,2}) = \rho_{2^a,2} i_{2^a} = i_2$. From (3) and (1), we obtain

$$2\rho_{2^a,2} = i_2\eta j_2 \rho_{2^a,2} = 2^{a-1} i_2\eta j_{2^a} = \begin{cases} i_2\eta j_2 & a = 1, \\ 0 & a > 1. \end{cases}$$

Therefore, the lemma is shown. □

By the above lemma, the relation $4 \cdot 1_{M(2)} = 0$ holds, and we have a retraction

(7) $$a\colon M(4) \wedge M(2) \to M(2)$$

of $i_4 \wedge 1_{M(2)}$, that is, $a(i_4 \wedge 1_{M(2)}) = 1_{M(2)}$.

Lemma 8 $a(1_{M(4)} \wedge i_2) \equiv \rho_{4,2} \mod (i_2\eta j_4)$.

Proof. By Lemma 6, we have $a(1_{M(4)} \wedge i_2) \in [M(4), M(2)] = \mathbb{Z}/2\{i_2\eta j_4\} \oplus \mathbb{Z}/2\{\rho_{4,2}\}$. Put

$$a(1_{M(4)} \wedge i_2) = x_2 i_2\eta j_4 + y_2\rho_{4,2}$$

where $x_2, y_2 \in \mathbb{Z}/2$, and $i_2 = a(i_4 \wedge 1_{M(2)})i_2 = a(1_{M(4)} \wedge i_2)i_4 = (x_2 i_2\eta j_4 + y_2\rho_{4,2})i_4 = y_2\rho_{4,2}i_4 = y_2 i_2$ by (1). This implies that $y_2 = 1$, and the lemma follows. □

Let BP be the Brown-Peterson spectrum at 2, and $BP_*(-)$ denotes the homology theory represented by BP. We then have $BP_* = BP_*(S^0) = \mathbb{Z}_{(2)}[v_1, v_2, \dots]$ and $BP_*(BP) = BP_*[t_1, t_2, \dots]$ with $|v_i| = |t_i| = 2^{i+1} - 2$. We also denote the unit map and the multiplication of BP by $\iota\colon S^0 \to BP$ and $\mu\colon BP \wedge BP \to BP$, respectively. The pair $(BP_*(X), BP_*(BP))$ is a Hopf algebroid for any spectrum X, and we have the Adams-Novikov spectral sequence

$$E_2^{s,t} = E_2^{s,t}(X) = \mathrm{Ext}_{BP_*(BP)}^{s,t}(BP_*, BP_*(X)) \Rightarrow \pi_{t-s}(X).$$

For the mod 16 Moore spectrum $M(16)$, the E_2-term $E_2^{0,8}(M(16))$ contains an element represented by $v_1^4 + 8v_1v_2 \in BP_8(M(16))$. Ravenel showed that this survives to $v \in \pi_8(M(16))$ (*cf.* [2, p.436]).

Proposition 9 The Adams v_1-periodic map $v_1^4\colon \Sigma^8 M(2) \to M(2)$ is the composite

$$\Sigma^8 M(2) \xrightarrow{v \wedge 1_{M(2)}} M(16) \wedge M(2) \xrightarrow{\rho_{16,4} \wedge 1_{M(2)}} M(4) \wedge M(2) \xrightarrow{a} M(2),$$

where a is in (7).

Proof. Put $x = a(\rho_{16,4}v \wedge 1_{M(2)})$ and consider the induced map $x_*\colon BP_*/(2) \to BP_*/(2)$. By Lemma 8 and (1), our claim follows from

(10) $$\begin{aligned} x_*(1) &= (1_{BP} \wedge x)(\iota \wedge i_2) \\ &= (1_{BP} \wedge a)(1_{BP} \wedge \rho_{16,4}v \wedge 1_{M(2)})(\iota \wedge i_2) \\ &= (1_{BP} \wedge a(1_{M(4)} \wedge i_2))(\iota \wedge \rho_{16,4}v) \\ &= (1_{BP} \wedge (\rho_{4,2} + x_2 i_2\eta j_4))(\iota \wedge \rho_{16,4}v) \\ &= (1_{BP} \wedge \rho_{4,2}\rho_{16,4})(\iota \wedge v) \\ &\quad + x_2(1_{BP} \wedge i_2\eta j_4\rho_{16,4})(\iota \wedge v) \\ &= (1_{BP} \wedge \rho_{16,2})(\iota \wedge v) \\ &\quad + 4x_2(1_{BP} \wedge i_2\eta j_{16})(\iota \wedge v) \\ &= \iota \wedge \rho_{16,2}v \\ &= (\iota \wedge \rho_{16,2})_*(v) \\ &= (\rho_{16,2})_*(v_1^4 + 8v_1v_2) \\ &= v_1^4. \end{aligned}$$

Therefore, x induced the multiplication by v_1^4. □

Recall that we have the map $\overline{\rho_{16,2}\widetilde{\rho}}\colon \Sigma^8 M(2) \to M(2)$ in (4).

Proposition 11 We may choose v as $\widetilde{\sigma}$.

Proof. From [2, p.429], we obtain $\pi_7(S^0) = \mathbb{Z}/16\{\alpha_{4/4}\}$. Therefore, by (2), $\alpha_{4/4} = c\sigma$ where $c \in (\mathbb{Z}/16)^\times$. On the other hand, the map v satisfies that $j_{16}v = \alpha_{4/4}$ by the geometric boundary theorem. Therefore, the relation $j_{16}v = \alpha_{4/4} = c\sigma$ holds. □

Proof of Theorem 5. Consider the induced homomorphism

$$\overline{\rho_{16,2}\widetilde{\sigma}}_*\colon BP_*/(2) \to BP_*(2).$$

Then, by Proposition 11 and (10), we have $\overline{\rho_{16,2}\widetilde{\sigma}}_*(1) = (1_{BP} \wedge \overline{\rho_{16,2}\widetilde{\sigma}})(\iota \wedge i_2) = \iota \wedge \overline{\rho_{16,2}\widetilde{\sigma}}i_2 = \iota \wedge \rho_{16,2}\widetilde{\sigma} = \iota \wedge \rho_{16,2}v = v_1^4$. Therefore, the theorem is proved. □

References

[1] H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies 49, Princeton Univ. Press, Princeton, 1962.

[2] D. C. Ravenel, A novice's guide to the Adams-Novikov spectral sequence, Geomtric applications of homotopy theory (Proc. Conf. Evanston, III, 1977), 404–475.

[3] D. C. Ravenel, *Nilpotence and Periodicity in stable homotopy theory*, Ann. of Math. Studies 128, Princeton Univ. Press, Princeton, 1992.