Toward the p-local homotopy of eo_{p-1}

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In [2], M. Hill calculated the homotopy groups of the connected higher real K-theory eo_4 at the prime 5. In this note, we consider the spectral sequence converging to the p-local homotopy groups of the spectrum eo_{p-1} at $p \geq 5$.

1. Introduction

Let p be a prime number and $K(n)$ the n-th Morava K-theory at p. Hopkins and Miller showed that any closed subgroup of the n-th Morava stabilizer group $\mathbb{G}_n$ acts on the n-th Morava E-theory spectrum E_n. They also showed that the homotopy fixed point spectrum $E_n^{h\mathbb{G}_n}$ is isomorphic to the $K(n)$-localized sphere spectrum $S^0_{K(n)}$. For any p and n, the group $\mathbb{G}_n$ has a finite maximal subgroup G_n. Gourvanov and Hopkins defined the *higher real K-theory*

$$EO_n = E_n^{hG_n}.$$

The inclusion $i\colon G_n \to \mathbb{G}_n$ induces

$$(1) \qquad S^0 \to S^0_{K(n)} = E_n^{h\mathbb{G}_n} \xrightarrow{i^*} E_n^{hG_n} = EO_n.$$

Under the composite, we expect that $\pi_*(EO_n)$ has much information of $\pi_*(S^0)$.

For example, at $(p,n) = (2,1)$, the algebra $\mathbb{G}_1$ is $\mathbb{Z}_2^\times$, the units of 2-adic integers, and $G_1 = \{\pm 1\} = C_2$, the cyclic group of order 2. In this case, the spectrum EO_1 is $E_1^{hC_2} = KO_2$, the 2-completed real K-theory. Therefore, the map (1) at $(p,n) = (2,1)$ is $S^0 \to KO_2$.

Hereafter, we consider the case that $p-1$ divides n. At $(p,n) = (2,1)$, the higher real K-theory EO_1 is KO_2. When $(p,n) = (3,2)$, the spectrum EO_2 is isomorphic to the $K(2)$-localization of TMF. These two higher real K-theories have connective models ko and tmf, that is, the $K(1)$-localization of ko is KO_2, and the $K(2)$-localization of tmf is $TMF_{K(2)}$. The homotopy groups $\pi_*(ko)$ are well known, and $\pi_*(tmf)$ was determined by Bauer [1]. In [2], Hill calculated the homotopy groups of the connective model eo_4 at $p = 5$ (see Theorem 8). Our hope is to generalize this result for the homotopy groups of eo_{p-1} at $p \geq 5$.

2. Spectral sequence converging to $\pi_*(eo_{p-1})$

For the fixed prime number p, we denote $q = 2(p-1)$. We consider the curve of the form

$$y^{p-1} = x^p + a_1x^{p-1} + \cdots + a_{p-1}x + a_p.$$

After the coordinate transformation $x \mapsto x + r$, we obtain

$$(2) \qquad y^{p-1} = x^p + \eta_R(a_1)x^{p-1} + \cdots + \eta_R(a_{p-1})x + \eta_R(a_p).$$

This gives rise to the following Hopf algebroid:

$$(3) \qquad (A, \Gamma) = \left(\mathbb{Z}_{(p)}[a_1, \ldots, a_p], A[r]\right)$$

with $|a_i| = iq$ and $|r| = q$. The left unit $\eta_L\colon A \to \Gamma$ and the coproduct $\Delta\colon \Gamma \to \Gamma \otimes_A \Gamma$ are given by

$$\eta_L(a_i) = a_i \quad \text{and} \quad \Delta(r) = r \otimes 1 + 1 \otimes r,$$

and the right unit $\eta_R\colon A \to \Gamma$ is defined by (2). This Hopf algebroid is called a *generalized Weierstrass Hopf algebroid*.

Example 4　At $p = 3$, we have

$$\begin{array}{rcl} y^2 = x^3 + a_1x^2 + a_2x + a_3 & & \\ \xrightarrow{x \mapsto x+r} \quad y^2 & = & (x+r)^3 + a_1(x+r)^2 + a_2(x+r) + a_3 \\ & = & x^3 + \underline{(a_1+3r)}x^2 + \underline{(a_2 + 2a_1r + 3r^2)}x \\ & & + \underline{(a_3 + a_2r + a_1r^2 + r^3)}, \end{array}$$

which implies that

$$\begin{aligned} \eta_R(a_1) &= a_1 + 3r, \\ \eta_R(a_2) &= a_2 + 2a_1r + 3r^2, \\ \eta_R(a_3) &= a_3 + a_2r + a_1r^2 + r^3. \end{aligned}$$

Theorem 5（Gorvanov-Hopkins-Mahowald）　For the spectrum eo_{p-1} at $p \in \{2,3\}$, the Adams-Novikov spectral sequence converging to $\pi_*(eo_{p-1})$ is of the form

$$E_2^{s,t}(eo_{p-1}) = \mathrm{Ext}^{s,t}_{(A,\Gamma)}(\Gamma, \Gamma) \Rightarrow \pi_{t-s}(eo_{p-1}).$$

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Remark 6 In [2], Hill assumed that the connected model eo_4 has the Adams-Novikov spectral sequence as above. Hereafter, we assume that the spectrum eo_{p-1} at arbitrary p satisfies the condition.

Remark 7 Even if the spectrum eo_{p-1} does not satisfy the condition, there exists an isomorphism

$$E_2^{*,*}(EO_{p-1}) = \mathrm{Ext}^{*,*}_{(A,\Gamma)}(\Gamma,\Gamma)[\Delta^{-1}]^\wedge_I$$

for some element Δ and ideal I.

3. Main result

First, we recall the following:

Theorem 8（Hill [2]） At $p = 5$, we have an isomorphism

$$E_2^{0,*}(eo_4) = \mathbb{Z}_{(5)}[c_2, c_3, \Delta_i, \Delta'_{15}, \Delta'_{18} : 4 \le i \le 22]/(\mathrm{rels})$$

where the degree of c_i, Δ_i and Δ'_i is $8i$. Furthermore,

$$E_2^{*,*}(eo_4) = E_2^0(eo_4)[a, b]/(\mathrm{rels})$$

where $|a| = (1, 8)$ and $|b| = (2, 40)$. The non-zero differentials are generated by

$$d_9(\Delta_{20}) = cab^4 \quad \text{and} \quad d_{33}(a\Delta_{20}^4) = c'b^{17},$$

where c and c' are in $\mathbb{Z}^\times_{(5)}$.

Hill's idea is the following: From (3), we obtain the Hopf algebroid

$$(\bar{A}, \bar{\Gamma}) = \left(\mathbb{Z}_{(p)}[a_1, \dots, a_{p-1}], \bar{A}[r]/(r^p + a_1 r^{p-1} + \cdots + a_{p-1} r)\right)$$

satisfying that

$$\mathrm{Ext}^{*,*}_{(\bar{A},\bar{\Gamma})}(\bar{A},\bar{A}) = \mathrm{Ext}^{*,*}_{(A,\Gamma)}(A, A).$$

The ideals $I_k = (p, a_1, \dots, a_k)$ of $\bar{A}$ are invariant and fit into

$$I_0 \subset I_1 \subset \cdots \subset I_{p-1} = \bar{A}.$$

Put

$$H^{*,*}_{(k)} = \mathrm{Ext}^{*,*}_{(\bar{A}/I_k, \bar{\Gamma}/I_k)}(\bar{A}/I_k, \bar{A}/I_k),$$

and we have the (a_k-)Bockstein spectral sequence

$$H^{*,*}_{(k)} \otimes \mathbb{Z}_{(p)}[a_k] \Rightarrow H^{*,*}_{(k-1)}.$$

Therefore, the structure of $E_2^{*,*}(eo_4)$ is calculated as follow:

$$H^{*,*}_{(4)} \Rightarrow H^{*,*}_{(3)} \Rightarrow H^{*,*}_{(2)} \Rightarrow H^{*,*}_{(1)} \Rightarrow H^{*,*}_{(0)} \Rightarrow E_2^{*,*}(eo_4).$$

Theorem 9（Hill [2]） Let $E(-)$ and $P(-)$ be exterior and polynomial algebras, respectively. For $p \ge 5$,

1. $H^{*,*}_{(p-1)} = E(a) \otimes P(b)$, where a is the cohomology class $\{r\}$ and b is the p-fold Massay product $\langle a, \dots, a\rangle$,
2. $H^{*,*}_{(p-2)} = E(a) \otimes P(a_{p-1}, b)$, and
3. $H^{*,*}_{(p-3)} = E(a) \otimes P(a_{p-1}, \Delta, b)\{x_1, \dots, x_{p-2}\}/(\mathrm{rels})$, where $\Delta = a_{p-1}^p$ and $x_i = \langle i!2^i a_{p-2}^i, \underbrace{a, \dots, a}_{i+1}\rangle$.

Theorem 10（Hill [2]） At $p = 5$, the non-zero differentials of the a_2-Bockstein spectral sequence

$$H^{*,*}_{(2)} \otimes \mathbb{Z}_{(5)}[a_2] = E(a) \otimes P(a_2, a_3, \Delta, b)\{x_1, x_2, x_3\}/(\mathrm{rels}) \Rightarrow H^{*,*}_{(1)}$$

are generated by

$$d_1(a_3) = 3a_2 a, \quad d_1(x_3) = 2a_2 a_3^2 b,$$
$$d_2(a_3^3) = -a_2^2 x_1 \quad \text{and} \quad d_2(x_2) = -a_2^2 b.$$

Furthermore,

$$H^{*,*}_{(1)} = E(a) \otimes P(a_2, a_3^2, a_3^5, \Delta, b)\{x_1\}/(\mathrm{rels}).$$

Theorem 11 Assume that $p \ge 5$. For the non-zero differentials of the a_{p-3}-Bockstein spectral sequence

$$\begin{aligned} &H^{*,*}_{(p-3)} \otimes \mathbb{Z}_{(5)}[a_{p-3}] \\ &\quad = E(a) \otimes P(a_{p-3}, a_{p-2}, \Delta, b)\{x_1, \dots, x_{p-2}\}/(\mathrm{rels}) \Rightarrow H^{*,*}_{(p-4)}, \end{aligned}$$

we have

$$d_1(a_{p-2}) = 3a_{p-3} a, \quad d_1(x_{p-2}) = -(p-2)!2^{p-2} a_{p-3} a_{p-2}^{p-3} b,$$
$$\text{and} \quad d_2(a_{p-2}^3) = -\frac{27}{2} a_{p-3}^2 x_1.$$

4. Conjectures

In this section, we use the notation

$$a \doteq b$$

if $a = cb$ for some $c \in \mathbb{Z}^\times_{(p)}$. By Theorem 10, we have

$$(12) \qquad d_2(x_2) \doteq a_2^2 b$$

at $p = 5$. We remark that Theorem 11 doesn't contain a generalization of this differential.

Conjecture 13 For the Bockstein spectral sequence in Theorem 11, the differential (12) is generalized to

$$d_{\frac{p-1}{2}}(x_{\frac{p-1}{2}}) \doteq a_{p-3}^{\frac{p-1}{2}} b.$$

Furthermore,

$$H^{*,*}_{(p-4)} = E(a) \otimes P(a_{p-3}, a_{p-2}^2, a_{p-2}^5, \Delta, b)\{x_1, \dots, x_{\frac{p-3}{2}}, x_{\frac{p+1}{2}}, \dots, x_{p-3}\}/(\mathrm{rels}).$$

We notice that the first Smith-Toda spectrum $V(1)$ exists at $p = 5$. Therefore, by Theorem 10, we have the Adams-Novikov spectral sequence

$$E_2^{*,*} = E(a) \otimes P(a_2, a_3^2, a_3^5, \Delta, b)\{x_1\}/(\mathrm{rels}) \Rightarrow \pi_*(eo_4 \wedge V(1)).$$

Theorem 14（Hill [2]） In the spectral sequence, the non-zero differentials are generated by

$$d_9(\Delta) \doteq ab^4, \quad d_{17}(x_1 \Delta^2) \doteq a_2 b^9,$$
$$d_{25}(a_2 \Delta^3) \doteq x_1 b^{12} \quad \text{and} \quad d_{33}(a\Delta^4) \doteq b^{17}.$$

At $p = 7$, the second Smith-Toda spectrum $V(3)$ exists. If Conjecture 13 is true, then we have

$$E_2^{*,*} = E(a)\otimes P(a_4, a_5^2, a_5^5, \Delta, b)\{x_1, x_2, x_4\}/(\text{rels}) \Rightarrow \pi_*(eo_6 \wedge V(3)).$$

Conjecture 15 At $p \in \{5, 7\}$,

$$d_{(p-3)q+1}(x_1\Delta^{p-3}) \doteq a_{p-3}b^{(p-2)^2} \quad \text{and} \quad d_{3q+1}(a_{p-3}\Delta^3) \doteq x_1 b^{3(p-1)}.$$

We notice that $(p-3)q+1 < 3q+1$ if $p = 5$, and $(p-3)q+1 > 3q+1$ if $p = 7$. Therefore, even if the conjecture holds, $E_{(p-3)q+2}$-term and E_{3q+2}-term have no general form for $p \geq 5$.

References

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[2] M. Hill, The 5-local homotopy of eo_4, Algebr. Geom. Topol. **8** (2008), 1741–1761.