A spectrum which is quasi E-equivalent to the sphere spectrum

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Let p be a prime number and $E(n)_*(-)$ the homology theory represented by the n-th Johnson-Wilson spectrum $E(n)$ at p. An $E(n)$-local spectrum X is an exotic sphere spectrum if $E(n)_*(X)$ is isomorphic to the $E(n)_*$-homology of the sphere spectrum as an $E(n)_*(E(n))$-comodule. Yosimura introduced the notion of quasi E-equivalence for a ring spectrum E. Kamiya and Shimomura proved that an $E(n)$-local spectrum X is an exotic sphere spectrum if and only if X belongs to a summand $\mathrm{Pic}^0(\mathcal{L}_n)$ of the Picard group of the $E(n)$-local stable homotopy category [1]. In this note, we show that any $E(n)$-local exotic sphere spectrum is quasi $E(n)$-equivalent to the sphere spectrum. In addition, we prove that an $E(0)$-local spectrum which is quasi $E(0)$-equivalent to the sphere spectrum is only the $E(0)$-localized sphere spectrum.

1. Introduction

Let $\mathcal{S}$ be the stable homotopy category of spectra, and S^k the k-dimensional sphere spectrum. For a spectrum E, we have the Bousfield localization functor $L_E\colon \mathcal{S} \to \mathcal{S}$ with respect to E. we denote $\mathcal{L}_E = L_E(\mathcal{S})$. Under the E-local smash product $L_E(- \wedge -)\colon \mathcal{L}_E \times \mathcal{L}_E \to \mathcal{L}_E$, the category $\mathcal{L}_E$ is a symmetric monoidal category with the unit object L_ES^0. An E-local spectrum X is *invertible in* $\mathcal{L}_E$ if there exists a spectrum $Y \in \mathcal{L}_E$ such that $L_E(X \wedge Y) = L_ES^0$. The *Picard group* $\mathrm{Pic}(\mathcal{L}_E)$ *of* $\mathcal{L}_E$ is defined to be the collection of isomorphism classes of invertible spectra. If the collection $\mathrm{Pic}(\mathcal{L}_E)$ is a set, then the E-local smash product defines a commutative group structure of $\mathrm{Pic}(\mathcal{L}_E)$, that is, $[X] + [Y] = [L_E(X \wedge Y)]$. For the sake of simplicity, we denote by X the class $[X] \in \mathrm{Pic}(\mathcal{L}_E)$ represented by X.

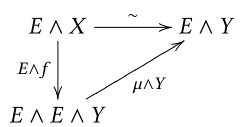
For a spectrum E, the *Bousfield class* $\langle E\rangle$ *of* E is defined by $\langle E\rangle = \{X \in \mathcal{S}\colon X \wedge E = *\}$, where $*$ is the point spectrum. We also define the order of Bousfield classes by $\langle E\rangle \geq \langle F\rangle \Leftrightarrow \langle E\rangle \subset \langle F\rangle$. As a well-known result, $\langle E\rangle \geq \langle F\rangle$ if and only if $L_EL_F = L_F = L_FL_E$, particularly, $L_F(\mathcal{L}_E) = \mathcal{L}_F$.

Let p be a prime number and $\mathcal{S}_{(p)}$ the stable homotopy category of p-local spectra. For any p and a non-negative integer n, we have the n-th Johnson-Wilson spectrum $E(n)$. Traditionally, we denote $L_n = L_{E(n)}$ and $\mathcal{L}_n = \mathcal{L}_{E(n)}$. The spectrum $E(0)$ is the rational Eilenberg-Mac Lane spectrum, and so we have $\mathrm{Pic}(\mathcal{L}_0) = \mathbb{Z}$ generated by L_0S^1. We know that if $n \geq m$, then $\langle E(n)\rangle \geq \langle E(m)\rangle$. In particular, for any $n \geq 0$, the relation $\langle E(n)\rangle \geq \langle E(0)\rangle$ holds. Therefore, since $L_0(\mathcal{L}_n) = \mathcal{L}_0$, the Bousfield localization functor L_0 induces the homomorphism $\ell_0\colon \mathrm{Pic}(\mathcal{L}_n) \to \mathrm{Pic}(\mathcal{L}_0) = \mathbb{Z}$ of groups. This homomorphism has a section, which assigns to an integer k the spectrum L_nS^k. Put $\mathrm{Pic}^0(\mathcal{L}_n) = \ker \ell_0$, and the group $\mathrm{Pic}(\mathcal{L}_E)$ admits a decomposition

(1) $$\mathrm{Pic}(\mathcal{L}_n) = \mathbb{Z} \oplus \mathrm{Pic}^0(\mathcal{L}_n),$$

where the summand $\mathbb{Z}$ is generated by L_nS^1. In [1], Kamiya and Shimomura showed that the isomorphism class of an $E(n)$-local spectrum X is in $\mathrm{Pic}^0(\mathcal{L}_n)$ if and only if the $E(n)_*$-homology of X is isomorphic to the $E(n)_*$-homology of S^0 as an $E(n)_*(E(n))$-comodule. We call such X an *exotic sphere spectrum* in $\mathcal{L}_n$.

Assume that E is a (unital, commutative and associative) ring spectrum, and we denote by $\iota\colon S^0 \to E$ and $\mu\colon E \wedge E \to E$ the unit map and the multiplication of E, respectively. A map $f\colon X \to E \wedge Y$ in $\mathcal{S}$ is a *quasi E-equivalence* if the composite $(\mu \wedge Y)(E \wedge f)$ is an isomorphism:



We say that X *is quasi E-equivalent to* Y if such $f\colon X \to E \wedge Y$ exists, and denote $X \sim_E Y$. The collection Θ_E is defined to be the collection of isomorphism classes of spectra which are quasi E-equivalent to the sphere spectrum. We also denote $\Theta_n = \Theta_{E(n)}$.

Theorem 2 $\mathrm{Pic}^0(\mathcal{L}_n) \subset \Theta_n$ for any p and n. Furthermore, if Θ_n is a set, then $\mathrm{Pic}^0(\mathcal{L}_n)$ is a subgroup of Θ_n.

Theorem 3 $\Theta_0 = \{L_0S^0\}$ for any p.

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We remark that, since L_0 is a smashing localization functor and $E(0)$ is a field spectrum, any $E(0)$-local spectrum is the point spectrum or a wedge sum of suspensions of $E(0)$, that is, any $X \in \mathcal{L}_0$ satisfies that

$$X = L_0X = X \wedge L_0S^0 = X \wedge E(0) = * \text{ or } \bigvee_i \Sigma^{k_i}E(0).$$

Theorem 3 claims that if an $E(0)$-local spectrum X satisfies that $X \sim_{E(0)} S^0$, then only such X is $E(0)$.

From theorems 2 and 3, we obtain another proof for $\mathrm{Pic}^0(\mathcal{L}_0) = \{L_0S^0\} = 0$, particularly, $\mathrm{Pic}(\mathcal{L}_0) = \mathbb{Z}$. We expected that the point of view yields another way for investigating Picard groups $\mathrm{Pic}(\mathcal{L}_n)$.

2. Properties of quasi E-equivalences

Let E be a ring spectrum whose structure is given by a unit map $\iota\colon S^0 \to E$ and a multiplication $\mu\colon E \wedge E \to E$.

Lemma 4 For any spectrum X, the relation $X \sim_E X$ holds.

Proof. For any spectrum X, the map $h = \iota \wedge X\colon X \to E \wedge X$ is a quasi E-equivalence. Indeed, $(\mu\wedge X)(E\wedge h) = (\mu\wedge X)(E\wedge\iota\wedge X)$ is the identity map of $E \wedge X$. Therefore, $X \sim_E X$. □

Proposition 5 Arbitrary X is quasi E-equivalent to L_EX.

Proof. For the canonical map $\eta\colon X \to L_EX$, the composite $h = (E \wedge \eta)(\iota \wedge X)\colon X \to E \wedge L_EX$ is a quasi E-equivalence. Indeed, we have $(\mu\wedge L_EX)(E\wedge h) = (\mu\wedge L_EX)(E\wedge E\wedge\eta)(E\wedge\iota\wedge X) = (E \wedge \eta)(\mu \wedge X)(E \wedge \iota \wedge X) = E \wedge \eta$, which is an isomorphism. □

Lemma 6 If $X \sim_E Y$, then $Y \sim_E X$.

Proof. Assume that $X \sim_E Y$, that is, there exists a quasi E-equivalence $f\colon X \to E \wedge Y$. Since $h = (\mu \wedge Y)(E \wedge f)\colon E \wedge X \to E \wedge Y$ is an isomorphism, we have an inverse h^{-1} of h. Then, the composite $h' = h^{-1}(\iota \wedge Y)\colon Y \to E \wedge X$ is a quasi E-equivalence. Indeed, we have

$$\begin{aligned}(\mu \wedge X)(E \wedge h') &= h^{-1}h(\mu \wedge X)(E \wedge h')\\ &= h^{-1}(\mu \wedge Y)(E \wedge f)(\mu \wedge X)(E \wedge h')\\ &= h^{-1}(\mu \wedge Y)(\mu \wedge E \wedge Y)(E \wedge E \wedge f)(E \wedge h')\\ &= h^{-1}(\mu \wedge Y)(E \wedge \underline{\mu \wedge Y})(E \wedge \underline{E \wedge f})(E \wedge h')\\ &= h^{-1}(\mu \wedge Y)(E \wedge \underline{h})(E \wedge h^{-1})(E \wedge \iota \wedge Y)\\ &= h^{-1}(\mu \wedge Y)(E \wedge \iota \wedge Y)\\ &= h^{-1}.\end{aligned}$$

Therefore, $Y \sim_E X$. □

For any spectrum X, we denote by EX the E-module spectrum $E \wedge X$ together with the structure $\mu \wedge X\colon E \wedge (E \wedge X) \to E \wedge X$.

Proposition 7 For two spectra X and Y, the following are equivalent:

1. $X \sim_E Y$.
2. $EX \simeq EY$ as an E-module spectrum.

Proof. Assume that there exists a quasi E-equivalence $f\colon X \to E \wedge Y$, that is, the composite $\bar{f} = (\mu \wedge Y)(E \wedge f)\colon E \wedge X \to E \wedge Y$ is an isomorphism. The map $\bar{f}$ satisfies that

$$\begin{aligned}\bar{f}(\mu \wedge X) &= (\mu \wedge Y)(E \wedge f)(\mu \wedge X)\\ &= (\mu \wedge Y)(\mu \wedge E \wedge Y)(E \wedge E \wedge f)\\ &= (\mu \wedge Y)(E \wedge \mu \wedge Y)(E \wedge E \wedge f)\\ &= (\mu \wedge Y)(E \wedge \bar{f}),\end{aligned}$$

and therefore $\bar{f}$ is an E-module isomorphism from EX to EY.

Conversely, we assume that there exists an E-module isomorphism $g\colon EX \to EY$. Then, the composite $g' = g(\iota \wedge X)\colon X \to E \wedge Y$ is a quasi E-equivalence. Indeed, $(\mu \wedge Y)(E \wedge g') = (\mu \wedge Y)(E \wedge g)(E \wedge \iota \wedge X) = g(\mu \wedge X)(E \wedge \iota \wedge X) = g$. □

We define

$$\Theta_E = \left\{X \in \mathcal{L}_E\colon S^0 \sim_E X\right\},$$

and denote by X the isomorphism class $[X] \in \Theta_E$.

Proposition 8 If the localization functor L_E is smashing, then the collection Θ_E is closed under the ordinarily smash product.

Proof. Since L_E is smashing, there exists a natural equivalence $L_E(-) \simeq (-) \wedge L_ES^0$. This implies that, for any X and Y are in $\mathcal{L}_E$, the smash product $X \wedge Y$ is in $\mathcal{L}_E$. Indeed, $X \wedge Y = X \wedge L_EY = X \wedge Y \wedge L_ES^0 = L_E(X \wedge Y)$. Therefore, it suffices to show that if $S^0 \sim_E X$ and $S^0 \sim_E Y$, then $S^0 \sim_E X \wedge Y$.

Assume that we have quasi E-equivalences $f\colon S^0 \to E \wedge X$ and $g\colon S^0 \to E \wedge X$. Consider the composite

$$h = (\mu \wedge X \wedge Y)(E \wedge T_{X,E} \wedge Y)(f \wedge g)\colon S^0 \to E \wedge X \wedge Y,$$

and

$$\begin{aligned}&(\mu \wedge X \wedge Y)(E \wedge h)\\ &= (\mu \wedge X \wedge Y)(E \wedge \mu \wedge X \wedge Y)(E \wedge E \wedge T_{X,E} \wedge Y)(E \wedge f \wedge g)\\ &= (\mu \wedge X \wedge Y)(\mu \wedge E \wedge X \wedge Y)(E \wedge E \wedge T_{X,E} \wedge Y)(E \wedge f \wedge g)\\ &= (\mu \wedge X \wedge Y)(E \wedge T_{X,E} \wedge Y)(E \wedge X \wedge g)(\mu \wedge X)(E \wedge f)\\ &= (E \wedge T_{Y,X})(\mu \wedge Y \wedge X)T_{X,E\wedge E\wedge Y}(T_{E,X} \wedge E \wedge Y)(E \wedge X \wedge g)(\mu \wedge X)(E \wedge f)\\ &= (E \wedge T_{Y,X})T_{X,E\wedge Y}(X \wedge \mu \wedge Y)(T_{E,X} \wedge E \wedge Y)(E \wedge X \wedge g)(\mu \wedge X)(E \wedge f)\\ &= (E \wedge T_{Y,X})T_{X,E\wedge Y}(X \wedge \underline{\mu \wedge Y})(X \wedge \underline{E \wedge g})T_{E,X}\underline{(\mu \wedge X)(E \wedge f)}\end{aligned}$$

is an isomorphism. Therefore, $S^0 \sim_E X \wedge Y$. □

For the E_*-homology theory $E_*(-) = \pi_*(E \wedge -)$, we denote $E_* = E_*(S^0) = \pi_*(E)$, the coefficient ring of $E_*(-)$. For any spectrum X, the E_*-homology $E_*(X)$ is an E_*-module, whose structure is induced by the E-module structure of EX.

Proposition 9 For an E-local spectrum X, the following are equivalent:

1. The spectrum X belongs to Θ_E.
2. $E_*(X) \cong E_*$ as an E_*-module.

Proof. Assume that $E_*(X) \cong E_*$ as an E_*-module. Then, $E_*(X) = E_*\{g\}$ for a generator $g \in E_0(X)$, and the generator g is a quasi E-equivalence from S^0 to $E \wedge X$. Indeed, the composite $\bar{g} = (\mu \wedge X)(E \wedge g)$ induces a homomorphism $\bar{g}_* = \pi_*(g)\colon E_* \to E_*(X)$ of E_*-modules. This homomorphism satisfies that $\bar{g}_*(\iota) = (\mu \wedge X)(E \wedge g)\iota = (\mu \wedge X)(\iota \wedge E \wedge X)g = g$. Therefore, $\bar{g}_*$ is an isomorphism, and so g is a quasi E-equivalence.

Conversely, we assume that X is in Θ_E. By Proposition 7, the relation $S^0 \sim_E X$ implies that $E = ES^0 \simeq EX$ as an E-module spectrum. Therefore, $E_* \cong E_*(X)$ as an E_*-module. □

3. Proof of main results

Consider the $E(n)$-local stable homotopy category $\mathcal{L}_n$ at a prime number p. For the Picard group $\mathrm{Pic}(\mathcal{L}_n)$, we have the decomposition $\mathrm{Pic}(\mathcal{L}_n) = \mathbb{Z} \oplus \mathrm{Pic}^0(\mathcal{L}_n)$ in (1). For the summand $\mathrm{Pic}^0(\mathcal{L}_n)$, Kamiya and Shimomura showed the following:

Theorem 10 ([1, Th. 1.1]) Let X be an $E(n)$-local spectrum. Then, X belongs to $\mathrm{Pic}^0(\mathcal{L}_n)$ if and only if $E(n)_*(X) = E(n)_*$ as an $E(n)_*(E(n))$-comodule.

Proof of Theorem 2. For any $X \in \mathrm{Pic}^0(\mathcal{L}_n)$, by Theorem 10, the $E(n)_*$-homology $E(n)_*(X)$ of X is isomorphic to $E(n)_*$ as an $E(n)_*(E(n))$-comodule. This implies that, by definition of $E(n)_*(E(n))$-comodule, $E(n)_*(X)$ is isomorphic to $E(n)_*$ as an $E(n)_*$-module. This follows that $X \in \Theta_n$ by Proposition 9.

Recall that the localization functor L_n is smashing, and this implies that $\mathrm{Pic}^0(\mathcal{L}_n)$ is closed under the ordinarily smash product. By Proposition 8, Θ_n is also closed under ordinarily smash product. Therefore, if Θ_n is a set, then the inclusion $\mathrm{Pic}^0(\mathcal{L}_n) \subset \Theta_n$ is a monomorphism of groups. □

Proof of Theorem 3. We know that the 0-th Johnson-Wilson spectrum $E(0)$ is the rational Eilenberg-Mac Lane spectrum, particularly, $L_0S^0 = E(0)$. If X belongs to Θ_0, then there exists a quasi $E(0)$-equivalence $S^0 \to E(0) \wedge X$. In particular, $E(0) = E(0) \wedge S^0 = E(0) \wedge X$. This implies that

$$X = L_0X = X \wedge L_0S^0 = X \wedge E(0) = E(0) = L_0S^0, \tag{11}$$

and therefore $\Theta_0 = \{L_0S^0\}$. □

Remark 12 The isomorphism (11) exists even if there exists an $E(0)$-equivalence $f\colon S^0 \to X$, that is, $E(0) \wedge f$ is an isomorphism. Therefore, we have

$$\Theta_E \subset \left\{X \in \mathcal{L}_0 \colon X \text{ is } E(0)\text{-equivalent to } S^0\right\}/\simeq \ = \{L_0S^0\}.$$

We expect that the condition $X \in \Theta_n$ implies more interesting facts.

References

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