Complex Cobordism and Eisentein Series

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In [1], Bodecker studied the Hirzebruch genus of level 3. In this note, we show that the image of the rationalization of the genus contains the Eisenstein series E_{2k} for an integer $k \geq 2$.

1 Introduction

Throughout this note, we denote $i = \sqrt{-1}$ and $q = e^{2\pi i z}$ for $z \in \mathbb{C}$. The formal power series associated with the Hirzeburch genus of level 3 is

$$Q^{\Gamma_1(3)}(x) = x\frac{\Phi\left(x - 2\pi i/3\right)}{\Phi(x)\Phi\left(-2\pi i/3\right)},$$

where

$$\Phi(x) = 2\mathrm{sinh}(x/2)\prod_{n=1}^{\infty}\frac{(1-e^{x}q^{n})(1-e^{-x}q^{n})}{(1-q^{n})^{2}}.$$

Let MU denote the complex cobordism spectrum, that is, $MU_* = \pi_*(MU)$ is generated by the cobordism classes of stably complex manifolds. In [1], Bodecker studied the Hirzeburch genus

$$\varphi^{\Gamma_1(3)} \colon MU_* \to M_*^{\Gamma_1(3)}, \tag{1.1}$$

where $M_*^{\Gamma_1(3)}$ is the graded ring of modular forms on $\Gamma_1(3)$. We recall that the infinite sum

$$G_k = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2}\frac{1}{(mz+n)^k} \quad \text{for} \quad z \in \mathbb{C} \tag{1.2}$$

is absolutely convergent if k is an even integer > 2. The Eisenstein series E_{2k} for $k \geq 2$ is defined by

$$E_{2k} = \frac{G_{2k}}{2\zeta(2k)}, \tag{1.3}$$

where $\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k}$, the Riemann zeta function. The main theorem in this paper is the following:

Theorem 1.4. *The image of*

$$\varphi^{\Gamma_1(3)} \otimes \mathbb{Q} \colon MU_* \otimes \mathbb{Q} \to M_*^{\Gamma_1(3)} \otimes \mathbb{Q}$$

contains any Eisenstein series E_{2k} for an integer $k \geq 2$.

By this result, we may consider that$MU_* \otimes \mathbb{Q}$ contains E_{2k}'s for $k \geq 2$. This is a new point of view for a relation between algebraic topology and number theory.

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2 R-valued genera

For the complex cobordism spectrum MU, the homotopy group $MU_* = \pi_*(MU)$ is the graded ring generated by the cobordism classes of stably complex manifolds. For a ring R, an *R-valued genus* is a ring homomorphism

$$\varphi \colon MU_* \to R.$$

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For a formal power series $Q(x) \in R[[x]]$, we consider $f_Q(x) = x/Q(x)$ and $g_Q(x) = f_Q^{-1}(x)$. The *genus associated with* $Q(x)$ is a homomorphism

$$\varphi_Q \colon MU_* \to R,$$

which sends the class $[\mathbb{C}P^n]$ of the n-dimensional complex projective space to the coefficient of x^n in $\frac{d}{dx}g_Q(x)$.

Lemma 2.1. *For $Q(x) = 1 + \sum_{n=1}^{\infty} a_n x^n \in R[[x]]$, the R-valued genus associated with $Q(x)$ satisfies*

$$\begin{aligned} \varphi_Q\left([\mathbb{C}P^1]\right) &= 2a_1, \text{ and} \\ \varphi_Q\left([\mathbb{C}P^3]\right) &= 4(a_1^3 + 3a_1a_2 + a_3). \end{aligned}$$

Proof. We note that

$$\begin{aligned} f_Q(x) &= \frac{x}{Q(x)} \\ &= x\left(1 - \sum_{n=1}^{\infty} a_n x^n + \left(\sum_{n=1}^{\infty} a_n x^n\right)^2 - \left(\sum_{n=1}^{\infty} a_n x^n\right)^3 + \cdots\right) \\ &= x\left(1 - a_1 x + (-a_2 + a_1^2)x^2 + (-a_3 + 2a_1a_2 - a_1^3)x^3 + \cdots\right) \\ &= x - a_1x^2 + (-a_2 + a_1^2)x^3 + (-a_3 + 2a_1a_2 - a_1^3)x^4 + \cdots. \end{aligned}$$

This implies that

$$g_Q(x) = f_Q^{-1}(x) = x + a_1x^2 + (a_1^2 + a_2)x^3 + (a_1^3 + 3a_1a_2 + a_3)x^4 + \cdots.$$

The lemma follows from this. □

3 Hirzeburch genus

Let M_k^{Γ} denote the group of modular forms of weight k on a group Γ, and

$$M_*^{\Gamma} = \bigoplus_k M_k^{\Gamma}.$$

For a positive integer N, we consider the group

$$\Gamma_1(N) = \left\{ A \in SL_2(\mathbb{Z}) \;\middle|\; A \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod (N) \right\}.$$

Definition 3.1. The *Hirzeburch genus of level* N is the $M_*^{\Gamma_1(N)}$-valued genus assocaited with the formal power series

$$Q^{\Gamma_1(N)}(x) = x\frac{\Phi(x - 2\pi i/N)}{\Phi(x)\Phi(-2\pi i/N)} \in M_*^{\Gamma_1(N)}[[x]],$$

where

$$\Phi(x) = 2\text{sinh}(x/2)\prod_{n=1}^{\infty}\frac{(1 - e^x q^n)(1 - e^{-x}q^n)}{(1 - q^n)^2}.$$

In the following, we consider the case $N = 3$.

Theorem 3.2 (Hornbostel-Neumann [3, Section 3.2]). *The graded ring $M_*^{\Gamma_1(3)}$ is the polynomial algebra $\mathbb{Z}[\omega, 1/3][E_1, E_3]$, where $\omega = e^{2\pi i/3}$. Here*

$$E_1 = 1 + 6\sum_{n=1}^{\infty}\sum_{0<d|n}\left(\frac{d}{3}\right)q^n \quad \textit{and} \quad E_3 = 1 - 9\sum_{n=1}^{\infty}\sum_{0<d|n}\left(\frac{d}{3}\right)d^2q^n,$$

where $\left(\frac{d}{3}\right)$ denotes the Legendre symbol.

Lemma 3.3 (Bodecker [1, p.2856]). *The formal power series $Q^{\Gamma_1(3)}(x)$ is of the form*

$$\begin{aligned} Q^{\Gamma_1(3)}(x) &= 1 + \frac{iE_1}{2\sqrt{3}}x + \frac{E_1^2}{12}x^2 + \frac{iE_1^3 - iE_3}{18\sqrt{3}}x^3 + \frac{13E_1^4 - 16E_1E_3}{2160}x^4 \\ &\quad + \frac{iE_1^2(E_1^3 - E_3)}{216\sqrt{3}}x^5 + \frac{121E_1^6 - 152E_1^3E_3 + 40E_3^2}{272160}x^6 \\ &\quad + \frac{iE_1(7E_1^6 - 11E_1^3E_3 + 4E_3^2)}{19440\sqrt{3}}x^7 + O(x^8). \end{aligned}$$

Consider the Hirzeburch genus

$$\varphi^{\Gamma_1(3)} = \varphi_{Q^{\Gamma_1(3)}} \colon MU_* \to M_*^{\Gamma_1(3)}. \tag{3.4}$$

From Lemma 2.1 and Lemma 3.3, we obtain

$$\begin{aligned} \varphi^{\Gamma_1(3)}\left([\mathbb{C}P^1]\right) &= 2\frac{iE_1}{2\sqrt{3}} = \frac{i}{\sqrt{3}}E_1 \quad \text{and} \\ \varphi^{\Gamma_1(3)}\left([\mathbb{C}P^3]\right) &= 4\left(-\frac{iE_1^3}{24\sqrt{3}} + \frac{iE_1^3}{8\sqrt{3}} + \frac{iE_1^3 - iE_3}{18\sqrt{3}}\right) = \frac{i}{9\sqrt{3}}(5E_1^3 - 2E_3). \end{aligned} \tag{3.5}$$

In $MU_* \otimes \mathbb{Q}$, the *Hazewinkel generators* v_i *at* 2 are defined by the following:

$$2^{n-1}v_n = [\mathbb{C}P^{2^n-1}] - \sum_{i=1}^{n-1} 2^{n-1-i} v_{n-i}^{2^i} [\mathbb{C}P^{2^i-1}]. \tag{3.6}$$

Lemma 3.7 (*cf.* Bodecker [1, p.2857])**.**

$$(\varphi^{\Gamma_1(3)} \otimes \mathbb{Q})(v_1) = \frac{i}{\sqrt{3}}E_1 \quad \textit{and} \quad (\varphi^{\Gamma_1(3)} \otimes \mathbb{Q})(v_2) = \frac{i}{9\sqrt{3}}(4E_1^3 - E_3).$$

Proof. From (3.5) and (3.6), we obtain

$$(\varphi^{\Gamma_1(3)} \otimes \mathbb{Q})(v_1) = (\varphi^{\Gamma_1(3)} \otimes \mathbb{Q})\left([\mathbb{C}P^1]\right) = \frac{i}{\sqrt{3}}E_1$$

and

$$\begin{aligned} (\varphi^{\Gamma_1(3)} \otimes \mathbb{Q})(v_2) &= (\varphi^{\Gamma_1(3)} \otimes \mathbb{Q})\left(\frac{[\mathbb{C}P^3] - v_1^2[\mathbb{C}P^1]}{2}\right) \\ &= (\varphi^{\Gamma_1(3)} \otimes \mathbb{Q})\left(\frac{[\mathbb{C}P^3] - [\mathbb{C}P^1]^3}{2}\right) \\ &= \frac{1}{2}\left(\frac{i}{9\sqrt{3}}\left(5E_1^3 - 2E_3\right) - \left(\frac{i}{\sqrt{3}}E_1\right)^3\right) = \frac{i}{9\sqrt{3}}(4E_1^3 - E_3). \end{aligned}$$

□

4 Main result

In [2], Borwein and Borwein introduced the following modular forms of level 3, which are called the *cubic theta functions*:

$$\begin{aligned} a(q) &= \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2}, \\ b(q) &= \sum_{m,n\in\mathbb{Z}} \omega^{n-m} q^{m^2+mn+n^2} \quad \text{and} \\ c(q) &= \sum_{m,n\in\mathbb{Z}} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2}, \end{aligned}$$

where ω is the complex number in Theorem 3.2.

Theorem 4.1 (Matsuda [4, Th. 1.1 and (1.8)])**.** *For* $q \in \mathbb{C}$ *with* $|q| < 1$,

$$a(q) = E_1 \quad \textit{and} \quad b(q)^3 = E_3.$$

Theorem 4.2 (Matsuda [4, Th. 5.1])**.** *For* $q \in \mathbb{C}$ *with* $|q| < 1$,

$$E_4 = 9a(q)^4 - 8a(q)b(q)^3 \quad \textit{and} \quad E_6 = -27a(q)^6 + 36a(q)^3b(q)^3 - 8b(q)^6.$$

By Theorem 3.2, these theorems imply the following:

Corollary 4.3. *The polynomial algebra* $\mathbb{Z}[\omega, 1/3]\,[E_4, E_6]$ *is a subalgebra of* $M_*^{\Gamma_1(3)}$.

Proof of Theorem 1.4. Consider the rationalization

$$\varphi^{\Gamma_1(3)} \otimes \mathbb{Q} \colon MU_* \otimes \mathbb{Q} \to M_*^{\Gamma_1(3)} \otimes \mathbb{Q}$$

of the homomorphism in (3.4). From Lemma 3.7, Theorem 4.1 and Theorem 4.2, we obtain

$$\begin{aligned}(\varphi^{\Gamma_1(3)} \otimes \mathbb{Q})(-207v_1^4 - 216v_1v_2) &= -207\left(\frac{i}{\sqrt{3}}E_1\right)^4 - 216\left(\frac{i}{\sqrt{3}}E_1\right)\left(\frac{i}{9\sqrt{3}}(4E_1^3 - E_3)\right)\\ &= 9E_1^4 - 8E_1E_3\\ &= 9a(q)^4 - 8a(q)b(q)^3\\ &= E_4,\end{aligned}$$

and

$$\begin{aligned}(\varphi^{\Gamma_1(3)} \otimes \mathbb{Q})(297v_1^6 + 2268v_1^3v_2 + 1944v_2^2) &= 297\left(\frac{i}{\sqrt{3}}E_1\right)^6 + 2268\left(\frac{i}{\sqrt{3}}E_1\right)^3\left(\frac{i}{9\sqrt{3}}(4E_1^3 - E_3)\right) + 1944\left(\frac{i}{9\sqrt{3}}(4E_1^3 - E_3)\right)^2\\ &= -27E_1^6 + 36E_1^3E_3 - 8E_3^2\\ &= -27a(q)^6 + 36a(q)^3b(q)^3 - 8b(q)^6\\ &= E_6.\end{aligned}$$

Since the polynomial algebra $\mathbb{Q}[E_4, E_6]$ contains any E_{2k} for an integer $k \geq 2$, the above calculation follows the theorem. □

Remark 4.4. By the above proof, for the subring $V = \mathbb{Q}[v_1, v_2]$ of $MU_* \otimes \mathbb{Q}$, the image of the restriction

$$\left(\varphi^{\Gamma_1(3)} \otimes \mathbb{Q}\right)\Big|_V : V \to M_*^{\Gamma_1(3)} \otimes \mathbb{Q}$$

contains any Eisenstein series E_{2k} for an integer $k \geq 2$. This implies that, for the homotopy group BP_* of the Brown-Peterson spectrum at the prime 2, the image of the homomorphism $BP_* \otimes \mathbb{Q} \to M_*^{\Gamma_1(3)} \otimes \mathbb{Q}$ induced by $\varphi^{\Gamma_1(3)} \otimes \mathbb{Q}$ contains any E_{2k} for an integer $k \geq 2$.

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