# The Algebra Structure of the Cohomology of $L(3,2)$

Ryo KATO*

Let $p$ be a prime number and $n$ an integer $\geq 0$. In [1], Ravenel introduced Lie algebras $L(n,k)$ in order to calculate the cohomology of the $n$-th Morava stabilizer algebra $S(n)$ at $p$. He determined the $\mathbb{Z}/p$-vector space structure of the cohomology of $S(3)$ at $p \geq 3$ by calculating the cohomology group of Lie algebras $L(3,1)$, $L(3,2)$ and $L(3,3)$. In this note, we determine the $\mathbb{Z}/p$-algebra structure of the cohomology of $L(3,2)$ for $p \geq 3$.

## 1 Introduction

Let $p$ be a prime number and $BP$ the Brown-Peterson spectrum at $p$. From the homology theory $BP_*(-)$ represented by $BP$, we obtain the Hopf algebroid

$$(BP_*, BP_*(BP)) = \left(\mathbb{Z}_{(p)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots]\right).$$

We also consider the algebra $K(n)_* = \mathbb{Z}/p[v_n^{\pm 1}]$. This is a $BP_*$-module, whose structure is given by $v_i v_n^s = v_n^{s+1}$ if $i = n$, and $= 0$ otherwise. Therefore, we obtain the Hopf algebra

$$(K(n)_*, \Sigma(n)) = (K(n)_* \otimes_{BP_*} BP_* \otimes_{BP_*} K(n)_*, K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*).$$

The $n$-th Morava stabilizer algebra is defined by

$$S(n) = \Sigma(n) \otimes_{K(n)_*} \mathbb{Z}/p,$$

where the $K(n)_*$-action on $\mathbb{Z}/p$ is given by $v_n \cdot 1 = 1$. In the following, we denote

$$H^*(A) = \mathrm{Ext}_A^*(\mathbb{Z}/p, \mathbb{Z}/p)$$

for a Hopf algebra $(\mathbb{Z}/p, A)$. The cohomology $H^*(S(n))$ is called the *cohomology of the $n$-th Morava stabilzer algebra*.

Consider the differential graded algebra

$$\Omega(n) = E(h_{i,j} \colon 1 \leq i \leq n,\ j \in \mathbb{Z}/n) \quad \text{with} \quad |h_{i,j}| = 1,$$

where $E(-)$ denotes an exterior algebra. The differential structure of $\Omega(n)$ is given by

$$d(h_{i,j}) = \sum_{1 \leq k < i} h_{k,j} h_{i-k,k+j}.$$

For the case $n < p-1$, there are two spectral sequences

$$E_2 = H^*\Omega(n) \Rightarrow H^*(E_0 S(n)) \quad \text{and} \quad E_2 = H^*(E_0 S(n)) \Rightarrow H^*(S(n))$$

by [1, Theorems 6.3.4, 6.3.5, and 6.3.8]. For the differential graded algebra $\Omega(n)$, we consider an increasing filtration

$$L(n,1) \subset L(n,2) \subset \cdots \subset L(n,n) = \Omega(n),$$

where

$$L(n,k) = E(h_{i,j} \colon 1 \leq i \leq k,\ j \in \mathbb{Z}/n).$$

Since $L(n,k)/L(n,k-1) = E(h_{k,j} \colon j \in \mathbb{Z}/n)$, we obtain the following spectral sequence:

$$H^*L(n,k-1) \otimes E(h_{k,j} \colon j \in \mathbb{Z}/n) \Rightarrow H^*L(n,k).$$

It is clear that $H^*L(n,1) = L(n,1) = E(h_{1,j} \colon j \in \mathbb{Z}/n)$, and we next turn to the cohomology of $L(n,2) = E\left(h_{1,j}, h_{2,j} \colon j \in \mathbb{Z}/n\right)$. The main theorem in this note is the following:

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* Faculty of Fundamental Science, National Institute of Technology (KOSEN), Niihama College, Niihama, 792-8580, Japan

**Theorem 1.1.** *As a $\mathbb{Z}/p$-algebra,*

$$H^*L(3,2) = \mathbb{Z}/p\,[h_j, g_j, k_j, e_j, K_j, l_j : j \in \mathbb{Z}/3]\,/R$$

*with $|h_j| = 1$, $|g_j| = |k_j| = |e_j| = 2$ and $|K_j| = |l_j| = 3$. Here $R$ is the ideal of the following relations: for $j, k \in \mathbb{Z}/3$,*

$$h_j h_k = 0, \quad e_0 + e_1 + e_2 = 0,$$
$$h_j g_j = h_j k_{j+2} = h_j e_{j+2} = 0, \quad h_j g_{j+1} = h_{j+2} k_j = h_{j+1} e_{j+1} = h_{j+1} e_{j+1}, \quad h_j g_{j+2} = h_{j+2} k_{j+2},$$
$$g_j^2 = g_j k_j = g_{j+1} k_j = k_j^2 = 0, \quad h_j K_k = \begin{cases} g_j g_{j+1} & k = j \\ -k_j k_{j+1} & k = j+1 \\ 0 & k = j+2 \end{cases}, \quad h_j l_k = \begin{cases} -k_{j+2} k_j & k = j \\ g_{j+1} k_{j+2} - g_j k_{j+1} & k = j+1 \\ g_{j+2} g_j & k = j+2 \end{cases},$$
$$g_j e_k = \begin{cases} 0 & k = j \\ -k_{j+2} k_j & k = j+1 \\ k_{j+2} k_j & k = j+2 \end{cases}, \quad k_j e_k = \begin{cases} g_j g_{j+1} & k = j \\ -g_j g_{j+1} & k = j+1 \\ 0 & k = j+2 \end{cases}, \quad e_j e_k = \begin{cases} -2 g_j k_{j+1} & k = j \\ g_{j+1} k_{j+2} - g_{j+2} k_j + g_j k_{j+1} & k = j+1 \\ k_{j+2} g_j - k_j k_{j+1} & k = j+2 \end{cases},$$
$$g_j K_j = g_j K_{j+2} = e_j K_k = 0,$$
$$k_j K_k = \begin{cases} g_{j+2} K_j & k = j+1 \\ 0 & k \neq j+1 \end{cases}, \quad g_j l_k = \begin{cases} g_{j+1} K_{j+2} & k = j+1 \\ 0 & k \neq j+1 \end{cases},$$
$$k_j l_k = \begin{cases} g_{j+1} K_{j+2} & k = j+2 \\ 0 & k \neq j+2 \end{cases}, \quad e_j l_k = \begin{cases} -2 g_{k+1} K_{k+2} & k = j+2 \\ g_{k+1} K_{k+2} & k \neq j+2 \end{cases},$$
$$K_j K_k = K_j l_k = l_j l_k = 0 \quad \textit{and} \quad g_0 g_1 g_2 = h_j g_{j+2} K_j = -k_0 k_1 k_2.$$

## 2 Proof of Theorem 1.1

It is clear that $H^0L(3,2) = \mathbb{Z}/p$. For $H^1L(3,2)$, we have the differentials

(2.1) $$d(h_{1,j}) = 0 \quad \text{and} \quad d(h_{2,j}) = h_{1,j} h_{1,j+1}$$

for $j \in \mathbb{Z}/3$. Therefore, we obtain the following:

**Lemma 2.2.** *The first cohomology of $L(3,2)$ is isomorphic to the $\mathbb{Z}/p$-vector space generated by*

$$h_{1,j} \quad \textit{for} \quad j \in \mathbb{Z}/3.$$

Next turn to $H^2L(3,2)$. For $j, k \in \mathbb{Z}/3$,

(2.3)
$$d(h_{1,j} h_{1,k}) = 0,$$
$$d(h_{1,j} h_{2,k}) = h_{1,j} h_{1,k} h_{1,k+1} = \begin{cases} \pm h_{1,0} h_{1,1} h_{1,2} & k = j+1 \\ 0 & \textit{otherwise} \end{cases} \quad \text{and}$$
$$d(h_{2,j} h_{2,j+1}) = h_{1,j} h_{1,j+1} h_{2,j+1} - h_{1,j+1} h_{1,j+2} h_{2,j}.$$

They give rise to the Massey products

$$g_j = \langle h_{1,j}, h_{1,j}, h_{1,j+1} \rangle = h_{1,j} h_{2,j},$$
$$k_j = \langle h_{1,j}, h_{1,j+1}, h_{1,j+1} \rangle = h_{2,j} h_{1,j+1} \quad \text{and}$$
$$e_j = \langle h_{1,j}, h_{1,j+1}, h_{1,j+2} \rangle = h_{1,j} h_{2,j+1} - h_{1,j+2} h_{2,j}.$$

From (2.1) and (2.3), we obtain the following lemma:

**Lemma 2.4.** *The second cohomology of $L(3,2)$ is isomorphic to the $\mathbb{Z}/p$-vector space generated by*

$$g_j,\ k_j,\ e_j \quad \textit{for} \quad j \in \mathbb{Z}/3.$$

*Furthermore, for $j, k \in \mathbb{Z}/3$, the following relations hold:*

$$h_{1,j} h_{1,k} = 0 \quad \textit{and} \quad e_0 + e_1 + e_2 = 0.$$

Next we notice the following differentials: for $j, k \in \mathbb{Z}/3$,

(2.5)
$$d(h_{1,0} h_{1,1} h_{1,2}) = 0,$$
$$d(h_{1,j} h_{1,j+1} h_{2,k}) = 0,$$
$$d(h_{1,j} h_{2,k} h_{2,k+1}) = \begin{cases} \pm h_{1,0} h_{1,1} h_{1,2} h_{2,j} & k = j \\ \pm h_{1,0} h_{1,1} h_{1,2} h_{2,j+2} & k = j+1 \\ 0 & k = j+2 \end{cases} \quad \text{and}$$
$$d(h_{2,0} h_{2,1} h_{2,2}) = h_{1,0} h_{1,1} h_{2,1} h_{2,2} + h_{1,1} h_{1,2} h_{2,2} h_{2,0} + h_{1,2} h_{1,0} h_{2,0} h_{2,1}.$$

By these differentials, the cohomology elements

$$K_j = \langle k_j, h_{1,j+1}, h_{1,j+2}\rangle = h_{2,j}h_{1,j+1}h_{2,j+1} \quad \text{and} \quad l_j = h_{1,j}h_{2,j}h_{2,j+1} + h_{1,j+1}h_{2,j+2}h_{2,j}$$

are defined.

**Lemma 2.6.** *The third cohomology of $L(3,2)$ is isomorphic to the $\mathbb{Z}/p$-vector space generated by*

$$h_{1,j}g_{j+1},\ h_{1,j}g_{j+2},\ K_j,\ l_j \quad \text{for} \quad j \in \mathbb{Z}/3.$$

*Furthermore, for $j \in \mathbb{Z}/3$, the following relations hold:*

$$\begin{aligned} &h_{1,j}g_j = h_{1,j}k_{j+2} = h_{1,j}e_{j+2} = 0,\\ &h_{1,j}g_{j+1} = h_{1,j+2}k_j = h_{1,j+1}e_{j+1} = h_{1,j+1}e_{j+2} \quad \text{and}\\ &h_{1,j}g_{j+2} = h_{1,j+2}k_{j+2}. \end{aligned} \tag{2.7}$$

*Proof.* For $j, k \in \mathbb{Z}/3$,

$$\begin{aligned} h_{1,j}g_j &= h_{1,j}\langle h_{1,j}, h_{1,j}, h_{1,j+1}\rangle = \langle h_{1,j}, h_{1,j}, h_{1,j}\rangle h_{1,j+1} = 0,\\ h_{1,j}k_{j+2} &= h_{1,j}\langle h_{1,j+2}, h_{1,j}, h_{1,j}\rangle = h_{1,j+2}\langle h_{1,j}, h_{1,j}, h_{1,j}\rangle = 0,\\ h_{1,j}g_{j+1} &= h_{1,j}(h_{1,j+1}h_{2,j+1}) \sim h_{1,j+1}h_{1,j+2}h_{2,j} = h_{1,j+2}k_{j+1},\\ h_{1,j}e_k &= h_{1,j}\langle h_{1,k}, h_{1,k+1}, h_{1,k+2}\rangle = \begin{cases} \langle h_{1,j}, h_{1,j}, h_{1,j+1}\rangle h_{1,j+2} & k = j\\ h_{1,j+1}\langle h_{1,j+2}, h_{1,j}, h_{1,j}\rangle & k = j+1\\ h_{1,j+2}\langle h_{1,j}, h_{1,j+1}, h_{1,j}\rangle & k = j+2 \end{cases}\\ &= \begin{cases} h_{1,j+2}g_j & k = j\\ h_{1,j+1}k_{j+2} & k = j+1\\ 0 & k = j+2 \end{cases},\\ h_{1,j}g_{j+2} &= h_{1,j}\langle h_{1,j+2}, h_{1,j+2}, h_{1,j}\rangle = h_{1,j+2}\langle h_{1,j+2}, h_{1,j}, h_{1,j}\rangle = h_{1,j+2}k_{j+2}. \end{aligned}$$

□

Next turn to $H^4L(3,2)$. We notice the following differentials: for $j \in \mathbb{Z}/3$,

$$\begin{aligned} &d(h_{1,0}h_{1,1}h_{1,2}h_{2,j}) = 0,\\ &d(h_{1,j}h_{1,j+1}h_{2,k}h_{2,k+1}) = 0 \quad \text{and}\\ &d(h_{1,j}h_{2,0}h_{2,1}h_{2,2}) = \pm h_{1,0}h_{1,1}h_{1,2}h_{2,j}h_{2,j+2}. \end{aligned} \tag{2.8}$$

**Lemma 2.9.** *The fourth cohomology of $L(3,2)$ is isomorphic to the $\mathbb{Z}/p$-vector space generated by*

$$g_jg_{j+1},\ g_jk_{j+1},\ k_jk_{j+1} \quad \text{for} \quad j \in \mathbb{Z}/3.$$

*Furthermore, for $j, k \in \mathbb{Z}/3$, the following relations hold:*

$$\begin{aligned} &g_j^2 = 0, \quad g_jk_j = 0, \quad g_{j+1}k_j = 0, \quad k_j^2 = 0,\\ &h_{1,j}K_k = \begin{cases} g_jg_{j+1} & k = j\\ -k_{j+1}k_{j+2} & k = j+1\\ 0 & k = j+2 \end{cases}, \quad h_{1,j}l_k = \begin{cases} -k_{j+2}k_j & k = j\\ g_{j+1}k_{j+2} - g_jk_{j+1} & k = j+1\\ g_{j+2}g_j & k = j+2 \end{cases},\\ &g_je_k = \begin{cases} 0 & k = j\\ -k_{j+2}k_j & k = j+1\\ k_{j+2}k_j & k = j+2 \end{cases}, \quad k_je_k = \begin{cases} g_jg_{j+1} & k = j\\ -g_jg_{j+1} & k = j+1\\ 0 & k = j+2 \end{cases} \quad \text{and}\\ &e_je_k = \begin{cases} -2g_jk_{j+1} & k = j\\ g_{j+1}k_{j+2} - g_{j+2}k_j + g_jk_{j+1} & k = j+1\\ k_{j+2}g_j - k_jk_{j+1} & k = j+2 \end{cases}. \end{aligned} \tag{2.10}$$

*Proof.* Immediately, we see that $g_j^2 = 0$, $g_jk_j = 0$, $g_{j+1}k_j = 0$ and $k_j^2 = 0$ for $j \in \mathbb{Z}/3$. The other relations are given as follows: for $j,k\in\mathbb{Z}/3$,

$$h_{1,j}K_k=h_{1,j}\langle k_k,h_{1,k+1},h_{1,k+2}\rangle=-k_k\langle h_{1,k+1},h_{1,k+2},h_{1,j}\rangle=\begin{cases}-k_je_{j+1} & k=j\\ -k_{j+1}k_{j+2} & k=j+1\\ 0 & k=j+2\end{cases},$$

$$\begin{aligned}h_{1,j}l_k&=h_{1,j}\left(h_{1,k}h_{2,k}h_{2,k+1}+h_{1,k+1}h_{2,k+2}h_{2,k}\right)\\&=h_{1,j}h_{1,k}h_{2,k}h_{2,k+1}+h_{1,j}h_{1,k+1}h_{2,k+2}h_{2,k}=\begin{cases}-k_{j+2}k_j & k=j\\ -g_{j+1}k_{j+2}+g_jk_{j+1} & k=j+1\\ g_{j+2}g_j & k=j+2\end{cases},\end{aligned}$$

$$g_je_k=(h_{1,j}h_{2,j})(h_{1,k}h_{2,k+1}-h_{1,k+2}h_{2,k})=\begin{cases}0 & k=j\\ -k_{j+2}k_j & k=j+1\\ k_{j+2}k_j & k=j+2\end{cases},$$

$$k_je_k=(h_{2,j}h_{1,j+1})(h_{1,k}h_{2,k+1}-h_{1,k+2}h_{2,k})=\begin{cases}g_jg_{j+1} & k=j\\ -g_jg_{j+1} & k=j+1\\ 0 & k=j+2\end{cases},$$

$$e_je_k=(h_{1,j}h_{2,j+1}-h_{1,j+2}h_{2,j})(h_{1,k}h_{2,k+1}-h_{1,k+2}h_{2,k})=\begin{cases}-2g_jk_{j+1} & k=j\\ g_{j+1}k_{j+2}-g_{j+2}k_j+g_jk_{j+1} & k=j+1\\ k_{j+2}g_j-k_jk_{j+1} & k=j+2\end{cases}.$$

□

For $H^5L(3,2)$, we have the following differentials: for $j\in\mathbb{Z}/3$,

(2.11) $$d(h_{1,0}h_{1,1}h_{1,2}h_{2,j}h_{2,j+1})=0 \quad\text{and}\quad d(h_{1,j}h_{1,j+11}h_{2,0}h_{2,1}h_{2,2})=0.$$

**Lemma 2.12.** *The fifth cohomology of $L(3,2)$ is isomorphic to the $\mathbb{Z}/p$-vector space generated by*

$$g_jK_{j+1}\quad\textit{for}\quad j\in\mathbb{Z}/3.$$

*Furthermore, for $j,k\in\mathbb{Z}/3$, the following relations hold:*

$$h_{1,j}g_jg_{j+1}=h_{1,j}g_{j+2}k_j=h_{1,j}k_jk_{j+1}=g_jK_j=g_jK_{j+2}=e_jK_j=0,$$
$$k_jK_k=\begin{cases}g_{j+2}K_j & k=j+1\\ 0 & k\neq j+1\end{cases},\quad g_jl_k=\begin{cases}g_{j+1}K_{j+2} & k=j+1\\ 0 & k\neq j+1\end{cases},$$
$$k_jl_k=\begin{cases}g_{j+1}K_{j+2} & k=j+2\\ 0 & k\neq j+2\end{cases}\quad\textit{and}\quad e_jl_k=\begin{cases}g_{k+1}K_{k+2} & k=j,j+1\\ -2g_{k+1}K_{k+2} & k=j+2\end{cases}.$$

*Proof.* The relations (2.7) imply the following: for $j,k\in\mathbb{Z}/3$,

$$\begin{aligned}h_{1,j}g_jg_{j+1}&=h_{1,j+1}e_{j+1}g_{j+1}=0,\\ h_{1,j}g_{j+2}k_j&=h_{1,j+2}k_{j+2}k_j=h_{1,j}g_{j+1}k_{j+2}=0,\\ h_{1,j}k_jk_{j+1}&=h_{1,j+1}k_jg_{j+2}=0,\\ g_jK_j&=(h_{1,j}h_{2,j})(h_{2,j}h_{1,j+1}h_{2,j+1})=0,\\ g_jK_{j+2}&=(h_{1,j}h_{2,j})(h_{2,j+2}h_{1,j}h_{2,j})=0,\\ k_jK_k&=(h_{2,j}h_{1,j+1})(h_{2,k}h_{1,k+1}h_{2,k+1})=\begin{cases}g_{j+2}K_j & k=j+1\\ 0 & k=j,j+2\end{cases},\\ e_jK_k&=(h_{1,j}h_{2,j+1}-h_{1,j+2}h_{2,j})(h_{2,k}h_{1,k+1}h_{2,k+1})=0,\\ g_jl_k&=(h_{1,j}h_{2,j})(h_{1,k}h_{2,k}h_{2,k+1}+h_{1,k+1}h_{2,k+2}h_{2,k})=\begin{cases}g_{j+1}K_{j+2} & k=j+1\\ 0 & k=j,j+2\end{cases},\\ k_jl_k&=(h_{2,j}h_{1,j+1})(h_{1,k}h_{2,k}h_{2,k+1}+h_{1,k+1}h_{2,k+2}h_{2,k})=\begin{cases}g_{j+1}K_{j+2} & k=j+2\\ 0 & k=j,j+1\end{cases},\\ e_jl_k&=(h_{1,j}h_{2,j+1}-h_{1,j+2}h_{2,j})(h_{1,k}h_{2,k}h_{2,k+1}+h_{1,k+1}h_{2,k+2}h_{2,k})=\begin{cases}g_{j+1}K_{j+2} & k=j\\ g_{j+2}K_j & k=j+1\\ -2g_jK_{j+1} & k=j+2\end{cases}.\end{aligned}$$

□

For the sixth cohomology, the differential

$$d(h_{1,0}h_{1,1}h_{1,2}h_{2,0}h_{2,1}h_{2,2}) = 0 \tag{2.13}$$

implies the following lemma:

**Lemma 2.14.** *The sixth cohomology of* $L(3,2)$ *is isomorphic to the* $\mathbb{Z}/p$*-vector space generated by the element*

$$g_0g_1g_2.$$

*Furthermore, for* $j,k \in \mathbb{Z}/3$ *the following relations hold:*

$$K_jK_k = K_jl_k = l_jl_k = 0 \quad \textit{and} \quad g_0g_1g_2 = h_{1,j}g_{j+2}K_j = -k_0k_1k_2. \tag{2.15}$$

*Proof.* Immediately, we see that $K_jK_k = 0$, $K_jl_k = 0$ and $l_jl_k = 0$ for $j,k \in \mathbb{Z}/3$. The relations (2.10) imply that

$$h_{1,j}g_{j+2}K_j = g_0g_1g_2 \quad \text{and} \quad k_0k_1k_2 = g_1e_0k_2 = -g_0g_1g_2$$

for $j \in \mathbb{Z}/3$. □

*Proof of Theorem 1.1.* The theorem follows from Lemmas 2.2, 2.4, 2.6, 2.9, 2.12 and 2.14. Here we replace $h_{1,j}$ with $h_j$. □

## References

[1] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, AMS Chelsea Publishing, Providence, 2004.