# A Note on the Multiplicative Structure of Ring Spectra

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For investigating a spectrum $R$, we have an important problem whether or not the spectrum $R$ has a multiplicative structure. The purpose of this note is to find a general condition whereby a left unital ring spectrum admits a two-sided, commutative, and associative multiplication.

## 1 Introduction

Throughout this note, for a spectrum $X$, we denote by $1_X$ the identity map of $X$.

**Definition 1.1.** A spectrum $R$ is a **left unital ring spectrum** if $R$ admits $\iota\colon S^0 \to R$ and $\mu\colon R \wedge R \to R$ satisfying

$$\mu(\iota \wedge 1_R) = 1_R.$$

These maps $\iota$ and $\mu$ are called the **unit map** and the **multiplication**, respectively.

**Definition 1.2.** Let $R$ be a left unital ring spectrum with a unit map $\iota$ and a multiplication $\mu$.

(i) The pair $(\iota, \mu)$ is **two-sided unital** if $\mu(1_R \wedge \iota) = 1_R$.
(ii) The pair $(\iota, \mu)$ is **commutative** if $\mu T = \mu$. Here $T$ is the switching map $R \wedge R \to R \wedge R$.
(iii) The pair $(\iota, \mu)$ is **associative** if $\mu(1_R \wedge \mu) = \mu(\mu \wedge 1_R)$.

Hereafter we consider a left unital ring spectrum $R$ with

(1.3) $$\iota\colon S^0 \to R \text{ and } \mu_0\colon R \wedge R \to R.$$

For the unit map $\iota$, we consider a cofiber sequence

(1.4) $$S^0 \xrightarrow{\iota} R \xrightarrow{\kappa} C \xrightarrow{\lambda} S^1.$$

We define

(1.5) $$\Xi_0 = \{\xi_0 \in [C, R]\colon \xi_0\kappa = 1_R - \mu_0(1_R \wedge \iota)\}.$$

Hereafter, for spectra $X$ and $Y$, $[X, Y]_n$ denotes the group of maps from $\Sigma^n X$ to $Y$. Beisdes $[X, Y]$ denotes $[X, Y]_0$.

**Theorem 1.6.** *There exists $\mu_1\colon R \wedge R \to R$ such that the pair $(\iota, \mu_1)$ is a two-sided unital ring spectrum structure if and only if $\xi_0(1_C \wedge \lambda) = 0$ for some $\xi_0 \in \Xi_0$.*

Suppose that $R$ admits a two-sided unital multiplication $\mu_1$. It is easy to see that, if $[R \wedge R, R]$ is 2-divisible, then

(1.7) $$\mu_2 = \frac{1}{2}(\mu_1 + \mu_1 T)$$

gives two-sided unital and commutative ring spectrum structure $(\iota, \mu_2)$ to $R$.

On the other hand, for the problem whether or not the pair $(\iota, \mu_1)$ is associative, we have the following result.

**Theorem 1.8.** *For the two-sided unital ring spectrum structure $(\iota, \mu_1)$, the associator is*

$$\alpha(\mu_1)(\kappa \wedge \kappa \wedge 1_R),$$

*where $\alpha(\mu_1)$ is in (2.3). Furthermore, the multiplication $\mu_1$ is associative if and only if $\alpha(\mu_1) = 0$.*

As an easy application, we apply these results to the cofiber of a generator in the homotopy groups of the sphere spectrum as follow:

**Corollary 1.9.** *Let $p$ be a prime number, and $f$ be an essential element in the homotopy group $\pi_n(S^0)$ of the $p$-local sphere spectrum. For a cofiber $W$ of $f$, the following hold:*

*(i) If $\pi_{2n+1}(W) = 0$, then $W$ has a two-sided unital ring spectrum structure.*
*(ii) In addition, if $p > 2$, then $W$ has a two-sided unital, and commutative ring spectrum structure.*

In the last section, as typical examples, we investigate the ring spectrum structure of

- The mod $p$ Moore spectrum, and
- The cofiber of the generator $\beta_1$ in the homotopy group $\pi_*(S^0)$, which is the first nontrivial element in the cokernel of $J$.

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## 2 Proof of Theorem 1.6 and Theorem 1.8

**Lemma 2.1.** *For a cofiber sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

*if $f$ has a retraction $r\colon Y \to X$, then there exists a unique map $\widehat{r}\colon Z \to Y$ such that*

*(i)* $g\widehat{r} = 1_Z$,

*(ii)* $fr + \widehat{r}g = 1_Y$, *and*

*(iii)* $r\widehat{r} = 0$.

*Proof.* Since $r$ is a retraction of $f$, for the cofiber sequence in the statement, we have $(1_Y - fr)f = 0$. This implies that there exists $\widehat{r}\colon Z \to Y$ such that

$$1_Y - fr = \widehat{r}g.$$

Besides $(g\widehat{r} - 1_Z)g = g(1_Y - fr) - g = 0$, and so we have a map $t\colon \Sigma X \to Z$ such that $th = g\widehat{r} - 1_Z$. Therefore,

$$g\widehat{r} - 1_Z = th = trfh = 0,$$

and

$$r\widehat{r} = rfr\widehat{r} = r(1_Y - \widehat{r}g)\widehat{r} = r\widehat{r}(1_Z - g\widehat{r}) = 0.$$

If there exists a map $\widehat{r}'$ such that $g\widehat{r}' = 1_Z$, $fr + \widehat{r}'g = 1_Y$, and $r\widehat{r}' = 0$, then

$$\widehat{r}' = \widehat{r}'g\widehat{r} = (1_Y - fr)\widehat{r} = \widehat{r}.$$

Therefore, the uniqueness of $\widehat{r}$ is shown. □

*Proof of Theorem 1.6.* First we consider a cofiber sequence

$$R \xrightarrow{\iota\wedge 1_R} R\wedge R \xrightarrow{\kappa\wedge 1_R} C\wedge R \xrightarrow{\lambda\wedge 1_R} \Sigma R. \tag{2.2}$$

We note that the map $\mu_0$ is a retraction of $\iota \wedge 1_R$. By Lemma 2.1, we have a map $\widehat{\mu}_0\colon C\wedge R \to R\wedge R$ such that

- $(\kappa\wedge 1_R)\widehat{\mu}_0 = 1_{C\wedge R}$,
- $(\iota\wedge 1_R)\mu_0 + \widehat{\mu}_0(\kappa\wedge 1_R) = 1_{R\wedge R}$, and
- $\mu_0\widehat{\mu}_0 = 0$.

Suppose that there exists $\mu_1$ in the statement, then

$$\begin{aligned}\mu_1\widehat{\mu}_0(1_C\wedge\iota)\kappa &= \mu_1\widehat{\mu}_0(\kappa\wedge 1_R)(1_R\wedge\iota)\\ &= \mu_1\left(1_{R\wedge R} - (\iota\wedge 1_R)\mu_0\right)(1_R\wedge\iota)\\ &= \mu_1(1_R\wedge\iota) - \mu_1(\iota\wedge 1_R)\mu_0(1_R\wedge\iota)\\ &= 1_R - \mu_0(1_R\wedge\iota).\end{aligned}$$

This implies that $\xi_0 = \mu_1\widehat{\mu}_0(1_C\wedge\iota)$ is in $\Xi_0$, and

$$\xi_0(1_C\wedge\lambda) = 0.$$

Conversely, we suppose that there exists $\xi_0 \in \Xi_0$ in the statement, and cosider a cofiber sequence

$$C \xrightarrow{1_C\wedge\iota} C\wedge R \xrightarrow{1_C\wedge\kappa} C\wedge C \xrightarrow{1_C\wedge\lambda} \Sigma C.$$

Since $\xi_0(1_C\wedge\lambda) = 0$, there exists a map $\xi_1\colon \Sigma C\wedge R \to R$ such that

$$\xi_1(1_C\wedge\iota) = \xi_0.$$

We define

$$\mu_1 = \mu_0 + \xi_1(\kappa\wedge 1_R),$$

then

$$\begin{aligned}\mu_1(\iota\wedge 1_R) &= \left(\mu_0 + \xi_1(\kappa\wedge 1_R)\right)(\iota\wedge 1_R)\\ &= \mu_0(\iota\wedge 1_R)\\ &= 1_R,\end{aligned}$$

and

$$\begin{aligned}\mu_1(1_R\wedge\iota) &= \left(\mu_0 + \xi_1(\kappa\wedge 1_R)\right)(1_R\wedge\iota)\\ &= \mu_0(1_R\wedge\iota) + \xi_1(\kappa\wedge 1_R)(1_R\wedge\iota)\\ &= \mu_0(1_R\wedge\iota) + \xi_1(1_C\wedge\iota)\kappa\\ &= 1_R - \xi_0\kappa + \xi_0\kappa \ \ (\text{by } \xi_0\in\Xi_0)\\ &= 1_R.\end{aligned}$$

Therefore, the pair $(\iota, \mu_1)$ is a two-sided unital ring spectrum structure on $R$. □

Next turn to Theorem 1.8. For a left unital multiplication $\mu\colon R\wedge R \to R$, we define

$$\alpha(\mu) = \mu(1_R\wedge\mu)(\widehat{\mu}\wedge 1_R)(1_C\wedge\widehat{\mu}). \tag{2.3}$$

*Proof of Theorem 1.8.* For a two-sided unital multiplication $\mu_1$, we have

$$\begin{aligned}&\mu_1(1_R\wedge\mu_1)(\widehat{\mu}_1\wedge 1_R)(\kappa\wedge 1_{R\wedge R})\\ =\ &\mu_1(1_R\wedge\mu_1)\left(1_{R\wedge R\wedge R} - (\iota\wedge 1_{R\wedge R})(\mu_1\wedge 1_R)\right)\\ =\ &\mu_1(1_R\wedge\mu_1) - \mu_1(\mu_1\wedge 1_R),\end{aligned}$$

and

$$\begin{aligned}&\mu_1(1_R\wedge\mu_1)(\widehat{\mu}_1\wedge 1_R)(1_C\wedge\iota\wedge 1_R)(1_C\wedge\mu_1)(\kappa\wedge 1_{R\wedge R})\\ =\ &\mu_1(1_R\wedge\mu_1)(\widehat{\mu}_1\wedge 1_R)(\kappa\wedge 1_{R\wedge R})(1_R\wedge\iota\wedge 1_R)(1_R\wedge\mu_1)\\ =\ &\mu_1(1_R\wedge\mu_1)(1_R\wedge\iota\wedge 1_R)(1_R\wedge\mu_1)\\ &-\mu_1(1_R\wedge\mu_1)(\iota\wedge 1_{R\wedge R})(\mu_1\wedge 1_R)(1_R\wedge\iota\wedge 1_R)(1_R\wedge\mu_1)\\ =\ &\mu_1(1_R\wedge\mu_1) - \mu_1(1_R\wedge\mu_1)\\ =\ &0.\end{aligned}$$

From these, the calculation

$$\begin{aligned}&\alpha(\mu_1)(\kappa\wedge\kappa\wedge 1_R)\\ =\ &\mu_1(1_R\wedge\mu_1)(\widehat{\mu}_1\wedge 1_R)(1_C\wedge\widehat{\mu}_1)(1_C\wedge\kappa\wedge 1_R)(\kappa\wedge 1_{R\wedge R})\\ =\ &\mu_1(1_R\wedge\mu_1)(\widehat{\mu}_1\wedge 1_R)(\kappa\wedge 1_{R\wedge R})\\ &-\mu_1(1_R\wedge\mu_1)(\widehat{\mu}_1\wedge 1_R)(1_C\wedge\iota\wedge 1_R)(1_C\wedge\mu_1)(\kappa\wedge 1_{R\wedge R})\\ =\ &\mu_1(1_R\wedge\mu_1) - \mu_1(\mu_1\wedge 1_R).\end{aligned}$$

follows. Therefore, if $\alpha(\mu_1) = 0$, then $\mu_1$ is associative. Conversely, if $\mu_1$ is associative, then

$$\alpha(\mu_1) = \left(\mu_1(1_R\wedge\mu_1) - \mu_1(\mu_1\wedge 1_R)\right)(\widehat{\mu}_1\wedge 1_R)(1_C\wedge\widehat{\mu}_1)$$

implies $\alpha(\mu_1) = 0$. □

# 3 Application

Let $p$ be a prime number. We take a nonzero element $f$ in the $n$-th homotopy group $\pi_n(S^0)$ of the $p$-local sphere spectrum, and consider a cofiber sequence

$$S^n \xrightarrow{f} S^0 \xrightarrow{i} W \xrightarrow{j} S^{n+1}. \tag{3.1}$$

**Lemma 3.2.** *There exists a map $\mu_0\colon W \wedge W \to W$ such that the pair $(i, \mu_0)$ is a left unital ring spectrum structure on $W$ if and only if $f \wedge 1_W \colon \Sigma^n W \to W$ is trivial.*

*Proof.* If the map $f \wedge 1_W \colon \Sigma^n W \to W$ is trivial, then there exists a retraction $\mu_0$ of $i \wedge 1_W$. Therefore $(i, \mu_0)$ is a left unital ring spectrum structure on $W$.

Conversely, if $(i, \mu_0)$ is left unital, then $\mu_0(i \wedge 1_W) = 1_W$. This implies $f \wedge 1_W = \mu_0(i \wedge 1_W)(f \wedge 1_W) = 0$. □

*Proof of Corollary 1.9.* Consider an exact sequence

$$\pi_{2n+1}(W) \xrightarrow{j^*} [W, W]_n \xrightarrow{i^*} \pi_n(W).$$

Since $\pi_{2n+1}(W) = 0$, the induced map $i^*$ is a monomorphism. We also have $i^*(f \wedge 1_W) = (f \wedge 1_W)i = if = 0$, and so the map $f \wedge 1_W$ is trivial. This implies that, by Lemma 3.2, the cofiber $W$ has a left unital ring spectrum structure. Remark that, in this case, the set $\Xi_0$ in (1.5) is of the form

$$\Xi_0 = \{\xi_0 \in \pi_{n+1}(W) \colon \xi_0 j = 1_W - \mu_0(1_W \wedge i)\}.$$

Since $\pi_{2n+1}(W) = 0$, we have $\xi_0 f = 0$ for any $\xi_0 \in \Xi_0$. Therefore, by Theorem 1.6, the cofiber $W$ admits a two-sided unital ring spectrum structure $(i, \mu_1)$. Furthermore, if $p > 2$, then the group $[W \wedge W, W]$ is 2-divisible, and it follows that $W$ admits a two-sided and commutative multiplication $\mu_2$ from (1.7). □

# 4 Examples

## 4.1 The mod $p$ Moore spectrum

Let $p$ be an odd prime number. The mod $p$ Moore spectrum $M(p)$ is defined by the cofiber sequence

$$S^0 \xrightarrow{p} S^0 \xrightarrow{i} M(p) \xrightarrow{j} S^1.$$

Recall that $\pi_0(S^0) = \mathbb{Z}_{(p)}$, $\pi_k(S^0) = 0$ for $k \in \{1, 2\}$, and

$$\pi_3(S^0) = \begin{cases} \mathbb{Z}/3\{\alpha_1\} & p = 3, \\ 0 & p > 3. \end{cases}$$

Therefore, by Corollary 1.9, the spectrum $M(p)$ admits a two-sided unital, and commutative ring spectrum structure $(i, \mu_2)$. We also note that $[M(p), M(p)] = \mathbb{Z}/p\{1_{M(p)}\}$. By Theorem 1.8, the multiplication $\mu_2$ is associative if and only if $\alpha(\mu_2) \in [M(p), M(p)]_2$ is trivial. Since

$$[M(p), M(p)]_2 = \begin{cases} \mathbb{Z}/3\{i\alpha_1 j\} & p = 3, \\ 0 & p > 3, \end{cases}$$

we have the following:

- If $p > 3$, then the mod $p$ Moore spectrum $M(p)$ is a two-sided unital, commutative, and associative ring spectrum.
- The mod 3 Moore spectrum $M(3)$ admits a two-sided unital, and commutative ring spectrum structure $(i, \mu_2)$. The associator of $\mu_2$ is of the form

  $$x \cdot i\alpha_1(j \wedge j \wedge j) \text{ with } x \in \mathbb{Z}/3.$$

  Indeed, by [4, Lemma 6.2], we know that $M(3)$ has no associative multiplication, and the associator is $\pm i\alpha_1(j \wedge j \wedge j)$.

**Remark 4.1.** In [1], Oka studied the ring spectrum structure of the mod $k$ Moore spectrum for an even integer $k$.

## 4.2 The cofiber of $\beta_1$

Let $p > 2$ and $q = 2(p-1)$. We consider $\beta_1 \in \pi_{pq-2}(S^0)$ and the cofiber sequence

$$S^{pq-2} \xrightarrow{\beta_1} S^0 \xrightarrow{i_W} W \xrightarrow{j_W} S^{pq-1}.$$

This induces an exact sequence

$$\pi_{2pq-3}(S^0) \xrightarrow{(i_W)_*} \pi_{2pq-3}(W) \xrightarrow{(j_W)_*} \pi_{pq-2}(S^0) \xrightarrow{(\beta_1)_*} \pi_{2pq-4}(S^0).$$

Recall that, by [3] and [2], we have

$$\pi_k(S^0) = \begin{cases} \mathbb{Z}/p^2\{\alpha_{p/2}\} & k = pq-1, \\ \mathbb{Z}/p\{\beta_1\} & k = pq-2, \\ 0 & k = 2pq-3, 2pq-2, 3pq-4, 3pq-3, \end{cases} \tag{4.2}$$

and $\beta_1^2 \neq 0$. They imply $\pi_{2pq-3}(W) = 0$, and so, by Corollary 1.9, the cofiber $W$ admits a two-sided unital, and commutative ring spectrum structure $(i, \mu_2)$. By Theorem 1.8, for the associativity of $\mu_2$, it suffices to investigate

$$\alpha(\mu_2) \in [W, W]_{2pq-2}.$$

Consider the following two exact sequences:

$$\pi_{3pq-3}(W) \xrightarrow{(j_W)^*} [W, W]_{2pq-2} \xrightarrow{(i_W)^*} \pi_{2pq-2}(W) \xrightarrow{(\beta_1)^*} \pi_{3pq-4}(W),$$

and

$$\pi_k(S^0) \xrightarrow{(i_W)_*} \pi_k(W) \xrightarrow{(j_W)_*} \pi_{k-pq+1}(S^0).$$

By these sequences and (4.2),

$$[W,W]_{2pq-2} = \mathbb{Z}/p^2\{\widetilde{\alpha}_{p/2}\} \text{ with } j_W\widetilde{\alpha}_{p/2}i_W = \alpha_{p/2},$$

and $\alpha(\mu_2) = x \cdot \widetilde{\alpha}_{p/2}$ with $x \in \mathbb{Z}/p^2$. By Theorem 1.8, the associator of $\mu_2$ is

$$x \cdot \widetilde{\alpha}_{p/2}(j_W \wedge j_W \wedge 1_W) \text{ with } x \in \mathbb{Z}/p^2.$$

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