# On a Simplified Method of Characterizing Hilbert Space

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The aim of this note is to inspect the simplified method of characterizing infinite-dimensional universal spaces given in the paper [1] for the case of complete spaces and proper maps. We adopt here a modified stability axiom for model spaces and give a detailed description of Toruńczyk's characterization theorem for Hilbert spaces following the flow of [1]. Detailed proofs of folklore statements are also given.

## 1 Introduction

All spaces under consideration are assumed to be separable metrizable and all maps continuous. The aim of this note is to inspect the simplified method of characterizing infinite-dimensional universal spaces proposed by Jan J. Dijkstra, Michael Levin and Jan van Mill [1]. In particular, we focus our attention to the case of complete spaces and proper maps. In section 2, we give detailed proofs of folklore statements called Edwards' shirinking and Edwards' trick. In sections 3–5, we give a detailed description of Toruńczyk's characterization theorem for Hilbert spaces following the flow of [1]. After inspecting proofs, we decided to adopt here the modified stability axiom $H \times \mathcal{Q} \approx H$ for a model space $H$ instead of the original one. For compact case, the original stability axiom $H \times [0,1] \approx H$ immediately implies $H \times \mathcal{Q} \approx H$ by Brown's approximation theorem [3, 6.7.4]. However, it is not so simple for non-compact case. Also, we give detailed proofs related to pseudo-interiors and the pseudo-boundaries of the Hilbert cube. We think they should be checked carefully though it is written in [1] simply.

## 2 Edwards' Trick

Let $f : X \to Y$ be a proper map and $A$ a subset of $X$. The mapping cylinder $M(f)$ of $f$ is considered as the quotient space of $X \times [0,1]$ replacing $X \times \{1\}$ with $Y$. We identify the space $M(f) \setminus Y$ with $X \times [0,1)$ and denote it by $X[0,1)$. In particular, we use the notation $X[0] = X$, $X[1] = Y$ and $X[0,1] = M(f)$. For each continuous map $\xi : X \to (0,1)$, we define $X[0,\xi] = \{(x,t) \in X[0,1) \mid 0 \le t \le \xi(x)\}$ and $X[\xi,1] = \{(x,t) \in X[0,1) \mid \xi(x) \le t\} \cup X[1]$. If $\eta : X \to (0,1)$ is a map with $\xi(x) \le \eta(x)$ for every $x \in X$, we define $X[\xi,\eta] = \{(x,t) \in X[0,1) \mid \xi(x) \le t \le \eta(x)\} \subset M(f)$. Similarly, the spaces $A[0,\xi]$, $A[\xi,\eta]$, $A[\xi,1]$ and so forth are defined in similar fashion.

Let $\beta : X \to Y$ be a map and $A \subset X$. We say that $\beta$ is *fully injective on $A$* if $\beta^{-1}(\beta(A)) = A$ and $\beta \restriction A : A \to Y$ is one-to-one.

**Proposition 2.1** (Edwards' Shrinking [2], [1])**.** *Let $f : X \to Y$ be a proper surjection between complete spaces. Let $\beta : M(f) \to Y$ be a proper map such that $\beta$ is fully injective on $X \times \{0\}$, where $c_f : M(f) \to Y$ is the collapsing map. Then, for each open cover $\mathscr{U}$ of $M(f)$, there is a homeomorphism $h : M(f) \to M(f)$ such that $c_f = c_f \circ h$ and the fibers of $\beta \circ h$ refines $\mathscr{U}$.*

*Proof.* Let $\mathscr{U}$ be an open cover of $M(f)$. Identifying $Y$ with $X[1] \subset X[0,1] = M(f)$, we take an open cover $\mathscr{U}_Y$ so that $\mathscr{U}_Y \prec \mathscr{U} \restriction Y$.

**Lemma 2.2.** *There are a map $\xi : X \to (0,1)$ and an open cover $\mathscr{V}$ of $Y$ such that $\{f^{-1}(V)[\xi,1] \mid V \in \mathscr{V}\} \prec \mathscr{U}$.*

*Proof.* For each $x \in X$, let $\xi'(x)$ be the infimum of $s \in [0,1]$ such that there exist a neighborhood $N$ of $f(x)$ in $Y$ and an elemtnt $U \in \mathscr{U}$ with $f^{-1}(N)[s,1] \subset U$. Then it is easy to see that $\xi' : X \to [0,1)$ is a well-defined function. Also, it is easy to see that $\xi'$ is an upper semi-continuous function.

Let $\xi : X \to (0,1)$ be a map such that $\xi(x) > 2^{-1}(1 + \xi'(x))$ for every $x \in X$. Fix a point $x \in X$ and let $\delta = \xi'(x)$ and $4\varepsilon = 1 - \delta$. Then there are open neighborhood $N$ of $f(x)$ and an element $U \in \mathscr{U}$ such that $f^{-1}(N)[\varepsilon + \delta, 1] \subset U$. The set $W = Y \setminus f(X \setminus \xi^{-1}((\varepsilon + \delta, 1]))$ is an open neighborhood of $f(x)$. Put $y = f(x)$ and $V_y = W \cap N$. Then the collection $\mathscr{V} = \{V_y \mid y \in Y\}$ is a required open cover of $Y$. □

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We may assume that $\mathscr{U}_Y \prec \mathscr{V}$ without loss of generality.

**Lemma 2.3.** *There exist an open cover $\mathscr{U}_X$ of $X$ and a sequence $\{t_i : X \to (0,1) \mid i \in \mathbb{N}\}$ of maps with $t_{i+1}(x) < t_i(x)$ and $\lim_{i\to\infty} t_i(x) = 0$, $\forall x \in X$, such that*

(1) $\{c_f^{-1}(U) \cap X[t_2, 1] \mid U \in \mathscr{U}_Y\} \prec \mathscr{U}$ *and*
(2) $\{U[t_{i+2}, t_i] \mid U \in \mathscr{U}_X\} \prec \mathscr{U}$, $\forall i \in \mathbb{N}$.

*Proof.* For each $x \in X$, we define $\gamma'(x)$ as the supremum of $s \in (0,1]$ such that, for each $t \in [0, 2^{-1}(\xi(x)+1)]$, there is an element $U \in \mathscr{U}$ such that $\{x\} \times ([t-s, t] \cap [0,1]) \subset U$. Then the function $\gamma'(x)$ is well-defined by the compactness of the interval $[0, 2^{-1}(\xi(x)+1)]$. Also, it follows that $\gamma'(x)$ is lower semi-continuous.

Let $\gamma : X \to (0,1)$ be a map such that $\gamma(x) < \min\{2^{-1}\gamma'(x),\ \xi(x)\}$ for every $x \in X$. Then we define the sequence of maps $t_1 > t_2 > t_3 > \cdots$ as follows: Define $t_2(x) = \xi(x)$ and $t_1(x) = \xi(x) + \min\{\gamma(x), 2^{-1}(1 - t_2(x))\}$. For $i \geq 3$, define $t_i(x) = t_{i-1}(x) - \min\{\gamma(x), 2^{-1}t_{i-1}(x)\}$. Then $t_{i+1}(x) < t_i(x)$ and $\lim_{i\to\infty} t_i(x) = 0$ for every $x \in X$. Since $c_f^{-1}(U) \cap X[t_2, 1] = f^{-1}(U)[t_2, 1]$ for every $U \in \mathscr{U}_Y$ and $\mathscr{U}_Y \prec \mathscr{V}$, the condition (1) is satisfied by Lemma 2.2. It is now obvious that $t_i(x) - t_{i+1}(x) < \gamma(x) < 2^{-1}\gamma'(x)$. By the definition of $\xi'$, each point $x \in X$ has an open neighborhood $U_x$ such that $U_x[t_{i+2}, t_i]$ is contained in some element $U_i \in \mathscr{U}$ for each $i \in \mathbb{N}$. Thus the open cover $\mathscr{U}_X = \{U_x \mid x \in X\}$ satisfies the condition (2). □

For each $y \in Y$, put $F_y = \beta^{-1}(y)$. Recall that $M(f) \setminus Y = X[0,1)$ and let $p_X : X[0,1) \to X$ be the projection.

**Lemma 2.4.** *There exists a sequence $\{s_i : X \to (0,1) \mid i \in \mathbb{N}\}$ of maps with $s_{i+1}(x) < s_i(x)$ and $\lim_{i\to\infty} s_i(x) = 0$, $\forall x \in X$, such that, for each non-degenerate fiber $F_y$,*

(1) *if $F_y \cap X[0, s_1] \neq \emptyset$ then $F_y \subset X[0,1)$ and $p_X(F_y) \subset U$ for some $U \in \mathscr{U}_X$,*
(2) *if $F_y \cap X[s_1, 1] \neq \emptyset$ then $F_y \subset X(s_2, 1]$,*
(3) *if $F_y \cap X[s_1, 1] = \emptyset$ then $F_y \subset X(s_{i+2}, s_i)$ for some $i \in \mathbb{N}$.*

*Proof.* For each $x \in X$, let $\beta_x$ denote the point $\beta((x,0))$. Let $U_x$ be an element of $\mathscr{U}_X$ such that $x \in U_x$ and let $V_x$ be an open neighborhood of $(x,0) \in M(f)$ such that $x \in V_x \subset p_X^{-1}(U_x) \subset X[0,1) \subset M(f)$. Since $\beta$ is a proper map and fully injective on $X[0]$, the point $\beta_x$ is not contained in the closed set $\beta(X[0,1] \setminus V_x)$. Hence we can take an open neighborhood $W_x$ of $\beta_x$ in $Y$ so that $W_x \subset Y \setminus \beta(X[0,1] \setminus V_x)$. Then $\beta^{-1}(W_x)$ is an open neighborhood of $x$ in $X[0,1]$ such that $\beta^{-1}(W_x) \subset p_X^{-1}(U_x) \subset X[0,1)$.

Put $W = \cup_{x\in X} W_x$. Then $W$ is an open neighborhood of $\beta(X[0])$ in $Y$ such that $\beta^{-1}(W) \subset X[0,1)$. Let $U$ be an open neighborhood of $\beta(X[0])$ in $Y$ such that $\beta(X[0]) \subset U \subset \overline{U} \subset W$. Put $G_1 = \beta^{-1}(U)$. Then $\beta^{-1}(\beta(G_1)) = G_1$ and $\overline{G_1} \subset \beta^{-1}(\overline{U}) \subset X[0,1)$. We define a function $s_1' : X \to (0,1)$ by $s_1'(x) = \sup\{s \in (0,1) \mid \{x\}[0,s] \subset G_1\}$. Now it is obvious that $s_1'$ is a well-defined and lower semi-continuous function.

Let $s_1 : X \to (0,1)$ be a map such that $s_1(x) < \min\{s_1'(x), 2^{-1}\}$ for every $x \in X$. Let $y = \beta_x \in Y$. If $F_y = \beta^{-1}(y)$ is a non-degenerate fiber with $F_y \cap X[0, s_1] \neq \emptyset$, then $F_y \subset \beta^{-1}(W) \subset X[0,1)$. Since $y = \beta_x \in W_x$ and $\beta^{-1}(W_x) \subset p_X^{-1}(U_x)$, we have $p_X(F_y) \subset U_x \in \mathscr{U}_X$, i.e., the condition (1) is satisfied.

By the same manner, we can take a neighborhood $G_2$ of $X[0]$ such that $\beta^{-1}(\beta(G_2)) = G_2$ and $\overline{G_2} \subset X[0, s_1)$. And we take the lower semi-continuous function $s_2' : X \to (0,1)$ defined by $s_2'(x) = \sup\{s \in (0,1) \mid \{x\}[0,s] \subset G_2\}$ and a map $s_2 : X \to (0,1)$ with $s_2(x) < \min\{s_2'(x), 2^{-2}\}$. Then the condition (2) is satisfied.

Inductively, as above, we construct an open neighborhood $G_i$ of $X[0]$ and a map $s_i : X \to (0,1)$ so that $\beta^{-1}(\beta(G_i)) = G_i$, $s_i < 2^{-i}$ and $X[0, s_i) \subset G_i \subset X[0, s_{i-1})$. Then the condition (3) is satisfied. □

Let $s_0,\ t_0 : X \to [0,1]$ be the constant map $s_0(x) = t_0(x) = 1$. Let $h' : X \times [0,1] \to X \times [0,1]$ be the homeomorphism that maps every interval $\{x\} \times [t_{i+1}(x), t_i(x)]$ linearly onto the corresponding interval $\{x\} \times [s_{i+1}(x), s_i(x)]$. Then we obtain the homeomorphism $h : M(f) \to M(f)$ induced by $h'$ sliding the $[0,1]$-factor, i.e., $c_f \circ h = c_f$.

Let $y \in Y$ and take an element $U_y \in \mathscr{U}_Y$ so that $y \in U_y \in \mathscr{U}_Y$. Now we suppose that the fiber $(\beta \circ h)^{-1}(y) = h^{-1}(F_y)$ is non-degenerate. If $F_y \cap X[s_1, 1] \neq \emptyset$ then $F_y \subset X(s_2, 1]$ by Lemma 2.4 (2). Then, by Lemma 2.3 (1), $h^{-1}(F_y) \subset c_f^{-1}(U_y)[t_2, 1]$ which is contained in some element of $\mathscr{U}$. In case $F_y \cap X[s_1, 1] = \emptyset$, we have $F_y \cap X[0, s_1] \neq \emptyset$, that is, there is an element $U \in \mathscr{U}_X$ such that $p_X(F_y) \subset U$ by Lemma 2.4 (1). Also, $F_y \subset X(s_{i+1}, s_i)$ for some $i \in \mathbb{N}$ by Lemma 2.4 (3). Then $h^{-1}(F_y) \subset U[t_{i+1}, t_i]$ which is contained in some element of $\mathscr{U}$ by Lemma 2.3 (2). Thus the fibers of $\beta \circ h$ refines $\mathscr{U}$. This completes the proof. □

**Lemma 2.5.** *If $f : X \to Y$ is a proper map then the collapsing map $c_f : M(f) \to Y$ is also a proper map.* □

**Corollary 2.6.** *If $f : X \to Y$ is a proper map and $\beta : M(f) \to Y$ is a map with $c_f = \beta \circ \alpha$ for some map $\alpha : M(f) \to M(f)$, then $\beta$ is a proper map.* □

**Theorem 2.7** (Bing's Shrinking Criterion (cf. [6, Theorem 2.7.1 and Remark 2.8]))**.** *A proper surjection $f : X \to Y$ between complete spaces is a near homeomorphism if it satisifies the following:*

(†) *For each open covers $\mathscr{U}$ of $X$ and $\mathscr{V}$ of $Y$, there is a homeomorphism $h : X \to X$ such that $f \circ h$ is $\mathscr{V}$-close to $f$ and $\{h(f^{-1}(y)) \mid y \in Y\} \prec \mathscr{U}$.* □

**Proposition 2.8** (Edwards' Trick [1, 2.1])**.** *Let $f : X \to Y$ be a proper surjection between complete spaces and let $c_f : M(f) \to Y$ be the collapsing map. Assume that, for each open cover $\mathscr{V}$ of $Y$, there are a near homeomorphism $\alpha : M(f) \to M(f)$ and a map $\beta : M(f) \to Y$ which is $\mathscr{V}$-close to $c_f$ and fully injective on $X \times \{0\}$ such that $c_f = \beta \circ \alpha$. Then $c_f$ is a near homeomorphism.*

*Proof.* Let $\mathscr{U}$, $\mathscr{U}_Y$ be open covers of $M(f)$ and $Y$ respectively. We shall construct a homeomorphism $h : M(f) \to M(f)$ which is $c_f^{-1}(\mathscr{U}_Y)$-close to the identity such that $\{(c_f \circ h)^{-1}(y) \mid y \in Y\} \prec \mathscr{U}$. Note that $c_f$ is a proper surjection by Lemma 2.5. Thus the proposition follows from the Bing's shrinking criterion 2.7.

Note that the map $\beta$ is proper by Corollary2.6. We refine $\mathscr{U}_Y$ so as to satisfy the condition of Lemma 2.3.

By Lemma 2.2 and Lemma 2.3, there exist an open cover $\mathscr{U}_X$ of $X$ and a sequence $\{t_i : X \to (0,1) \mid i \in \mathbb{N}\}$ of maps with $t_{i+1}(x) < t_i(x)$ and $\lim_{i\to\infty} t_i(x) = 0$, $\forall x \in X$, such that

(1) $\{c_f^{-1}(U_Y) \cap X[t_2, 1] \mid U_Y \in \mathscr{U}_Y\} \prec \mathscr{U}$ and
(2) $\{U_X[t_{i+2}, t_i] \mid U_X \in \mathscr{U}_X\} \prec \mathscr{U}$, $\forall i \in \mathbb{N}$.

Let $\mathscr{U}'_X$ be a star refinement of $\mathscr{U}_X$. By Lemma 2.4, there exists a sequence $\{s_i : X \to (0,1) \mid i \in \mathbb{N}\}$ of maps with $s_{i+1}(x) < s_i(x)$ and $\lim_{i\to\infty} s_i(x) = 0$, $\forall x \in X$, such that, for each non-degenerate fiber $F_y$,

(3) if $F_y \cap X[0, s_1] \neq \emptyset$ then $F_y \subset X[0,1)$ and $p_X(F_y) \subset U$ for some $U \in \mathscr{U}'_X$,
(4) if $F_y \cap X[s_1, 1] \neq \emptyset$ then $F_y \subset X(s_2, 1]$,
(5) if $F_y \cap X[s_1, 1] = \emptyset$ then $F_y \subset X(s_{i+2}, s_i)$ for some $i \in \mathbb{N}$.

Let $\mathscr{V}$ be an open neighborhood of $Y$ such that $\mathrm{st}\,\mathscr{V} \prec \mathscr{U}_Y$. Then there are a near homeomorphism $\alpha : M(f) \to M(f)$ and a map $\beta : M(f) \to Y$ which is $\mathscr{V}$-close to $c_f$ and fully injective on $X \times \{0\}$ such that $c_f = \beta \circ \alpha$. Let $\mathscr{W}$ be an open cover of $M(f)$ satisfying the following:

(6) $\{\mathrm{st}\,(F_y, \mathrm{st}\,\mathscr{W}) \mid y \in Y\} \prec c_f^{-1}(\mathscr{U}_Y)$,
(7) $\mathrm{st}\,\{W \in \mathscr{W} \mid W \subset X[0,1)\} \prec p_X^{-1}(\mathscr{U}'_X)$ and
(8) $\mathrm{st}\,(X[s_{i+1}, s_i], \mathrm{st}\,\mathscr{W}) \subset X[s_{i+2}, s_{i-1}]$, $\forall i \in \mathbb{N}$.

Let $\varphi : M(f) \to M(f)$ be a homeomorphism $\mathscr{W}$-close to $\alpha$. Let $\psi : M(f) \to M(f)$ be the sliding homeomorphism which maps the intervals $\{x\} \times [t_{i+1}, t_i] \to \{x\} \times [s_{2i+2}, s_{2i}]$ and $\{x\} \times [t_1, 1] \to \{x\} \times [s_2, 1]$ linearly for every $x \in X$ with $c_f \circ \psi = c_f$. We define $h : M(f) \to M(f)$ by $h = \varphi^{-1} \circ \psi$. By the condition (1)–(8), one can see that $\{(c_f \circ h)^{-1}(y) \mid y \in Y\} \prec \mathscr{U}$ and that $c_f \circ h$ is $\mathscr{U}_Y$-close to $c_f$. Thus the proposition follows from the Bing's shrinking criterion 2.7. □

## 3 Nice Maps

The Hilbert cube, the pseudo-interior and the pseudo-bandary are denoted by $\mathcal{Q}$, $\mathfrak{s}$ and $B(\mathcal{Q})$ respectively, that is, $\mathcal{Q} = \Pi_{i=1}^{\infty}[-1,1]_i$, $\mathfrak{s} = \Pi_{i=1}^{\infty}(-1,\ 1)_i$, and $B(\mathcal{Q}) = \mathcal{Q} \setminus \mathfrak{s}$. A surjective proper map $f : X \to Y$ is called a *cell-like map* if $f^{-1}(y)$ is a cell-like compactum for every $y \in Y$. A completely metrizable space is called an *$L$-space* if it is strongly universal $AR$.

**Definition 3.1.** An $L$-space $H$ is called an *$\ell_2$-model space* if it satisfies the following:

- (Stability) $H \approx H \times \mathcal{Q}$;
- ($Z$-set unknotting) For each open cover $\mathscr{U}$ of $H$ and a homeomorphism $h : Z_1 \to Z_2$ between $Z$-sets $Z_1$ and $Z_2$ of $H$ which is supported on some open set $U$ in $H$ and $\mathscr{U}$-homotopic to the identity, there exists a homeomorphism $\widetilde{h} : H \to H$ which is $\mathscr{U}$-homotopic to the identity such that $\widetilde{h} \restriction Z_1 = h$ and supported by $U$.

One should note that $H \approx H \times [0,1]$ whenever $H$ is an $\ell_2$-model space since $H \approx H \times \mathcal{Q} \approx H \times \mathcal{Q} \times [0,1] \approx H \times [0,1]$.

**Lemma 3.2.** *Let $Y$ be a space and $C$ a compactum. For each open cover $\mathscr{U}$ of $Y \times C$, there exists a map $\gamma : Y \to (0,1)$ such that $\{B((y,c), \gamma(y)) \mid (y,c) \in Y \times C\} \prec \mathscr{U}$.*

*Proof.* Let $r : Y \times C \to (0,1)$ be a map such that $\{B((y,c), r((y,c))) \mid (y,c) \in Y \times C\} \prec \mathscr{U}$ (c.f [5, 2.7.7(2)]). For each $y \in Y$, let $\xi(y) = \min\{r(y,c) \mid c \in C\}$. Then the function $\xi : Y \to (0,1)$ is well-defined and lower semi-continuous by the compactness of $C$. Hence we can take a map $\gamma : Y \to (0,1)$ such that $\gamma(y) < \xi(y)$

for every $y \in Y$. Then the map $\gamma$ satisfies the required condition. □

**Theorem 3.3** (Mapping Replacement Theorem (cf. [6, 3.1.12])). *If a complete ANR space $X$ is strongly universal then $X$ satisfies the followig: For ach open cover $\mathscr{U}$ of $X$, each complete space $Y$ and a map $f : Y \to X$ such that the restriction $f \upharpoonright A : A \to X$ to a closed subspace $A$ is a $Z$-embedding, there is a $Z$-embedding $g : Y \to X$ which is an extension of $f \upharpoonright A$ and $\mathscr{U}$-close to $f$.* □

**Theorem 3.4** ([5, 7.5.4]). *Let $f : X \to Y$ be a proper map between ANRs $X$ and $Y$. Then $f$ is a cell-like map if and only if $f$ is a fine homotopy equivalence.* □

For a proper map $f : X \to Y$ and a closed set $A \subset Y$, $X \cup_f A$ denotes the quotient space of $X$ obtained by collapsing the fibers over $A$ to singletons.

**Proposition 3.5** ([1, 3.1]). *Let $H$ be an $\ell_2$-model space, $Y$ an complete AR space and $A$ a closed subset of $Y$. If $f : H \to Y$ is a cell-like map such that $f^{-1}(A)$ is a $Z$-set in $H$ then the quotient map $\pi : H \to H \cup_f A$ is a near homeomorphism and $A$ is a $Z$-set in $H \cup_f A$.*

*Proof.* We shall show that $\pi : H \to H \cup_f A$ is a near homeomorphism by Bing's shrinking criterion 2.7. Let $\mathscr{U}$ and $\mathscr{V}$ be open covers of $H$ and $H \cup_f A$ respectively. Since $f$ is a proper map, we can take a collection $\mathscr{T}$ of open sets in $Y$ such that $\cup\mathscr{T} \supset A$ and $f^{-1}(\mathscr{T}) \prec \pi^{-1}(\mathscr{V})$. Let $\mathscr{S}$ be an open cover of $Y$ which refines the open cover $\mathscr{T} \cup \{Y \setminus A\}$ of $Y$. Since $f$ is a fine homotopy equivalence (Theorem 3.4), there is a map $g : Y \to H$ such that $g \circ f$ is $f^{-1}(\mathscr{S})$-homotopic to $1_H$. Let $\mathscr{U}'$ be an open star refinement of both $\mathscr{U}$ and $f^{-1}(\mathscr{S})$ such that any $\mathscr{U}'$-close two maps are $f^{-1}(\mathscr{S})$-homotopic. Using the strong universality of $H$, we obtain a $Z$-embedding $h : f^{-1}(A) \to H$ such that $h$ is $\mathscr{U}'$-close to $g \circ f$. Then $h$ is $f^{-1}(\mathscr{S})$-homotoic to the identity and $h(f^{-1}(a)) \subset \mathrm{st}(g(f(f^{-1}(a))), \mathscr{U}') = \mathrm{st}(g(a), \mathscr{U}')$ for each $a \in A$. Hence we have $\{h(f^{-1}(a)) \mid a \in A\} \prec \mathscr{U}$. By the $Z$-set unknotting of $H$, we obtain a homeomorphism $\widetilde{h} : H \to H$ supported by $f^{-1}(\cup\mathscr{T})$ such that $\widetilde{h}$ is $f^{-1}(\mathscr{S})$-close to $1_H$ with $\widetilde{h} \upharpoonright f^{-1}(A) = h \upharpoonright f^{-1}(A)$. Since $\{\widetilde{h}(f^{-1}(a)) \mid a \in A\} = \{h(f^{-1}(a)) \mid a \in A\} \prec \mathscr{U}$ and the non-degenerate fibers of $\pi$ is the set $\{f^{-1}(a) \mid a \in A\}$, we have $\{h(\pi^{-1}(x)) \mid x \in H \cup_f A\} \prec \mathscr{U}$. Hence the quotient map $\pi$ is a near homeomorphism by the Bing's shrinking criterion 2.7. Using a homeomorphism $H \to H \cup_f A$ sufficiently close to $\pi$, one can see that $A$ is a $Z$-set in $H \cup_f A$. □

**Corollary 3.6** ([1, 3.2]). *Let $H$ be an $\ell_2$-model space, $f : H \to Y$ a cell-like map and $Y$ a complete AR space. Then the quotient map $\pi : H \times [0,1] \to M(f)$ is a near homeomorphism. In particular, $H$ is homeomorphic to $M(f)$.*

*Proof.* Note that the map $f \times \mathrm{id} : H \times [0,1] \to Y \times [0,1]$ is cell-like and $Y \times [0,1]$ is a complete $AR$ space. Let $A = Y \times \{1\}$ and note that $(H \times [0,1]) \cup_{f\times\mathrm{id}} A = M(f)$. Then $A$ is a closed set in $H \times [0,1]$ and the set $(f \times \mathrm{id})^{-1}(A) = H \times \{1\}$ is a $Z$-set in $H \times [0,1]$ (cf. Lemma 3.2). Hence the quotient map $\pi : H \times [0,1] \to (H \times [0,1]) \cup_{f\times\mathrm{id}} A = M(f)$ is a near homeomorphism by Proposition 3.5. By the stability of $H$, we have $H \approx H \times [0,1] \approx M(f)$. □

**Lemma 3.7.** *Let $f : X \to Y$ be a proper map and let $C$ be a compactum. Then a map $g = (g_Y, g_C) : X \to Y \times C$ is proper if and only if the map $g_Y : X \to Y$ is proper.* □

**Lemma 3.8.** *Let $Y$ be a complete space and let $f = (f_Y, f_\mathcal{Q}) : X \to Y \times \mathcal{Q}$ be a map. For each open cover $\mathscr{U}$ of $Y \times \mathcal{Q}$, there exists a one-to-one maps $\alpha$, $\beta : X \to \mathcal{Q}$ such that the maps $f_\alpha = (f_Y, \alpha)$, $f_\beta = (f_Y, \beta) : X \to Y \times \mathcal{Q}$ are $\mathscr{U}$-close to $f$ with $f_\alpha(X) \subset Y \times B(\mathcal{Q})$ and $f_\beta(X) \subset Y \times \mathfrak{s}$.*

*Proof.* Let $f = (f_Y, f_\mathcal{Q}) : X \to Y \times \mathcal{Q}$ be a map and let $\mathscr{U}$ be an open cover of $Y \times \mathcal{Q}$. By Lemma 3.2, we take a continuous map $\gamma : Y \to (0,1)$ so that $\{B((y,c), \gamma(y)) \mid (y,c) \in Y \times C\} \prec \mathscr{U}$. For each $i \in \mathbb{N}$, let $Y_i = \{y \in Y \mid \gamma(y) \geq 2^{-i}\}$ and $X_i = f_Y^{-1}(Y_i)$. Since $\alpha$ is continuous, $X_i$ and $Y_i$ are closed in $X$ and $Y$ respectively. In particular, the family $\{X_i\}_{i=1}^\infty$ satisfies the conditions $X_1 \subset \mathrm{int}\, X_2 \subset X_2 \subset \cdots \subset X_{i-1} \subset \mathrm{int}\, X_i \subset X_i \subset \cdots$ and $X = \cup_{i=1}^\infty X_i$.

Let $X_0 = \emptyset$ and $f_0 = f_\mathcal{Q}$. We shall inductively construct a sequence of maps $f_i : X \to \mathcal{Q}$ satisfying the following:

(1) $f_i \upharpoonright X_{i-1} = f_{i-1} \upharpoonright X_{i-1}$,
(2) $f_i \upharpoonright X \setminus \mathrm{int}\, X_{i+1} = \mathrm{id}$,
(3) $f_i \upharpoonright X_i$ is one-to-one,
(4) $\widehat{d}(f_i,\ f_{i-1}) < 2^{-i-2}$ and
(5) $\overline{f_i(X_i)}$ is a $Z$-set in $\mathcal{Q}$ with $\overline{f_i(X_i)} \subset B(\mathcal{Q})$,

where $\overline{f_i(X_i)}$ is the closure of $f_i(X_i)$ in $\mathcal{Q}$.

Assume $f_{i-1} : X \to \mathcal{Q}$ has been constructed. Let $K_{i-1}$ and $K_i$ be the closures of $f_{i-1}(K_{i-1})$ and $f_{i-1}(K_i)$ in $\mathcal{Q}$ respectively, that is, $K_{i-1} = \overline{f_{i-1}(X_{i-1})}$ and $K_i = \overline{f_{i-1}(X_i)}$. Since $K_{i-1}$ is a $Z$-set in $\mathcal{Q}$, there exists a $Z$-embedding

$g : K_i \to \mathcal{Q}$ such that $g \restriction K_{i-1} = \mathrm{id}$ and $2^{-i-3}$-homotopic to the identity [4, 5.3.11]. Since $B(\mathcal{Q})$ is a capset in $\mathcal{Q}$ and $K_{i-1}$ is a $Z$-set contained in $B(\mathcal{Q})$, there exists a homeomorphism $h \in \mathcal{H}(\mathcal{Q})$ such that $h \restriction g(K_{i-1}) = h \restriction K_{i-1} = \mathrm{id}$, $h(g(K_i)) \subset B(\mathcal{Q})$ and $h$ is $2^{-i-3}$-homotopic to the identity [4, 5.4.2]. Since $h \circ g : K_i \to \mathcal{Q}$ is a $Z$-embedding which is $2^{-i-2}$-homotopic to the identity, we have a $2^{-i-2}$-homotopy $H : \mathcal{Q} \times [0,1] \to \mathcal{Q}$, $H_0 = \mathrm{id}$ and $H_1 = h \circ g$. Let $F : X \times [0,1] \to \mathcal{Q}$ be the pull-back of $H$ by $f_{i-1}$, that is, the map defined by $F(x,t) = H(f_{i-1}(x),t)$. Then $F$ is a $2^{-i-2}$-homotopy such that $F_0 = f_{i-1}$ and $F_1 = h \circ g \circ f_{i-1}$. Since $X_i$ and $X \setminus \mathrm{int} X_{i+1}$ are disjoint closed sets in $X$, there is a Urysohn map $\lambda : X \to [0,1]$ such that $\lambda(X_i) = \{1\}$ and $\lambda(X \setminus \mathrm{int} X_{i+1}) = \{0\}$. Then the map $f_i : X \to \mathcal{Q}$ defined by $f_i(x) = F(x, \lambda(x))$, $x \in X$ is a required one.

Now we consider the uniform limit $\alpha = \lim_{i\to\infty} f_i : X \to \mathcal{Q}$. Obviously, $\alpha$ is a well-defined continuous map. For $x \in X_i \setminus X_{i-1}$, we note that $f_{i-2}(x) = f_{\mathcal{Q}}(x)$ and the point moves at most twice by $f_{i-1}$ and $f_i$ until the limit point $\alpha(x)$. Hence, for $x \in X_i \setminus X_{i-1}$, we have $d(x, \alpha(x)) = d(x, f_i(x)) \le d(f_{i-2}(x), f_{i-1}(x)) + d(f_{i-1}(x), f_i(x)) \le 2^{-i-1} + 2^{-i-2} < 2^{-i}$. Moreover, it follows from (3) that $\alpha : X \to Q$ is a one-to-one map. It is obvious that the map $f_\alpha = (f_Y, \alpha) : X \to Y \times \mathcal{Q}$ satisfies the condition $f_\alpha(X) \subset Y \times B(\mathcal{Q})$. Also, we can see that $f_\alpha$ is $\mathscr{U}$-close to $f$ since $\widehat{d}(\alpha \restriction X_i \setminus X_{i-1},\ \mathrm{id}) < 2^{-i}$.

To construct a map $\beta : X \to Q$ with the property that $f_\beta = (f_Y, \beta) : X \to Y \times \mathcal{Q}$ is $\mathscr{U}$-close to $f$ and $f_\beta(X) \subset Y \times \mathfrak{s}$, we only need slight modifications of the arguments above. We consider the pseudo-interior $\mathfrak{s}$ instead of $B(\mathcal{Q})$, and construct a sequence of maps $\{f_i\}$ satisfying conditions (1)–(4) and

(5)′ $\overline{f_i(X_i)}$ is a $Z$-set in $\mathcal{Q}$ with $\overline{f_i(X_i)} \subset \mathfrak{s}$,

In the inductive step, we use the property of the pseudo-interior $\mathfrak{s}$ insted of $B(\mathcal{Q})$ to obtain a homeomorphism $h \in \mathcal{H}(\mathcal{Q})$ such that $h \restriction g(K_{i-1}) = h \restriction K_{i-1} = \mathrm{id}$, $h(g(K_i)) \subset \mathfrak{s}$ and $h$ is $2^{-i-3}$-homotopic to the identity [4, 5.3.5]. Then we can obtain the uniform limit map $\beta = \lim_{i\to\infty} f_i : X \to \mathcal{Q}$ which satisfies the required conditions. □

**Corollary 3.9.** *Let $X$ and $Y$ be complete spaces. For each proper map $f : X \to Y \times Q$ and an open cover $\mathscr{U}$ of $Y \times \mathcal{Q}$, there are closed embeddings $f_\alpha$, $f_\beta : X \to Y \times \mathcal{Q}$ which are $\mathscr{U}$-close to $f$ with $f_\alpha(X) \subset Y \times B(\mathcal{Q})$ and $f_\beta(X) \subset Y \times \mathfrak{s}$.*

*Proof.* We define $f_\alpha$ and $f_\beta$ as in Lemma 3.8. Then the maps $f_\alpha$ and $f_\beta$ are closed embedding by Lemma 3.7 since $f$ is a proper map. □

Let $f : X \to Y$ be a map and let $c_f : M(f) \to Y$ be the collapsing map. We say $f$ is a *nice map* if, for each open cover $\mathscr{U}$ of $Y$, there is a closed embedding $g : Y \to Y$ such that $g$ is $\mathscr{U}$-close to the identity and $c_f^{-1}(g(Y))$ is a $Z$-set in $M(f)$.

**Proposition 3.10.** *For a proper map $f : X \to Y$ between complete spaces $X$ and $Y$, the map $f \times 1_{\mathcal{Q}} : X \times \mathcal{Q} \to Y \times \mathcal{Q}$ is a nice map.*

*Proof.* First we note that the mapping cylinder $M(f \times 1_{\mathcal{Q}})$ can be written as $M(f) \times \mathcal{Q}$. And the collapsing map $c_{f \times 1_{\mathcal{Q}}} : M(f \times 1_{\mathcal{Q}}) \to Y \times Q$ is written by $c_f \times 1_{\mathcal{Q}} : M(f) \times \mathcal{Q} \to Y \times \mathcal{Q}$, where $c_f : M(f) \to Y$ is the collapsing map. Then the statement follow by Corollary 3.9. □

**Proposition 3.11** ([1, 3.3])**.** *Let $H$ be an $\ell_2$-model space and $Y$ an $L$-space. If $f : H \to Y$ is a nice cell-like map then the collapsing map $c_f : M(f) \to Y$ is a near homeomorphism.*

*Proof.* We use the notation $H[0] = H \times \{0\} \subset M(f) = H[0,1]$ as in section 2. To see that $c_f$ is a near homeomorphism, we shall use the Edwards' strategy 2.8. Let $\mathscr{V}$ and $\mathscr{V}'$ be open covers of $Y$ such that $\mathrm{st}^2\mathscr{V}' \prec \mathscr{V}$. Since $f$ is a nice map and $Y$ is strongly universal, there is a closed embedding $g : H[0] \to Y$ which is $\mathscr{V}'$-close to $c_f \restriction H[0]$ such that $c_f^{-1}(\mathrm{Im}\, g)$ is a $Z$-set in $M(f)$. By Corollary 3.6, $M(f)$ is homeomorphic to $H$. Let $A = \mathrm{Im}\, g \subset Y$. Since $f$ is a cell-like map, $c_f : M(f) \to Y$ is a cell-like map such that $c_f^{-1}(A)$ is a $Z$-set in $M(f)$. Then the quotient map $\pi : M(f) \to M(f) \cup_{c_f} A$ is a near homeomorphism and $A$ is a $Z$-set in $M(f) \cup_{c_f} A$ by Proposition 3.5. Let $q_f : M(f) \cup_{c_f} A \to Y$ be the projection and let $\mathscr{U}$ be an open star refinement of $q_f^{-1}(\mathscr{V}')$. Take a homeomorphism $h : M(f) \to M(f) \cup_{c_f} A$ which is $\mathscr{U}$-close to $\pi$. Let $g' : H[0] \to A \subset M(f) \cup_{c_f} A$ be the embedding induced by $g$. Thus, $q_f \circ g' = g$ and $g'$ is $q_f^{-1}(\mathscr{V}')$-close to $\pi \restriction H[0]$. Then $h^{-1} \circ g' : H[0] \to h^{-1}(A)$ is a homeomorphism between $Z$-sets in $M(f) \approx H$ which is $c_f^{-1}(\mathrm{st}\,\mathscr{V}')$-close to the identity. Indeed, $c_f \circ h^{-1} \circ g' = q_f \circ \pi \circ h^{-1} \circ g'$ is $\mathscr{V}'$-close to $q_f \circ h \circ h^{-1} \circ g' = q_f \circ g'$ which is $\mathscr{V}'$-close to $q_f \circ \pi = c_f$. Using $Z$-set unknotting, there is a homeomorphism $\gamma : M(f) \to M(f)$ which is $c_f^{-1}(\mathrm{st}\,\mathscr{V}')$-close to the identity such that $\gamma \restriction H[0] = h^{-1} \circ g'$. Then $h \circ \gamma : M(f) \to M(f) \cup_{c_f} A$ is a homeomorphism which is $q_f^{-1}(\mathrm{st}^2\mathscr{V}')$-close to the identity such that $h \circ \gamma(H[0]) = A$.

Put $\alpha = (h \circ \gamma)^{-1} \circ \pi : M(f) \to M(f)$ and $\beta = q_f \circ h \circ \gamma : M(f) \to Y$. Then $\beta \circ \alpha = q_f \circ \pi = c_f$ and $\beta$ is $\mathscr{V}$-close to $c_f$ since $\mathrm{st}^2 \mathscr{V}' \prec \mathscr{V}$. Also we have $\beta(H[0]) = q_f(A) = A$. Then $\beta$ is fully injective on $H[0]$ since $q_f^{-1}(A) = A$ and $h \circ \gamma$ is a homeomorphism with $h \circ \gamma \restriction H[0] = g' : h[0] \to A$. Thus $c_f$ is a near homeomorphism by Proposition 2.8. □

**Lemma 3.12.** *Let $X$ be a strongly universal complete space and $C$ a compactum. Then the projection $p_X : X \times C \to X$ is a nice map.*

*Proof.* Let $c_{p_X} : M(p_X) \to X$ be the collapsing map and note that $M(p_X) \approx X \times \mathrm{cone}(C)$. Let $f : X \to X$ be a $Z$-embedding sufficiently close to the identity. Then one can see that the set $c_{p_X}^{-1}(f(X)) \approx f(X) \times \mathrm{cone}(C)$ is a $Z$-set in $M(p_X) \approx X \times \mathrm{cone}(C)$ using Lemma 3.2. □

**Corollary 3.13.** *The projection $\pi_H : H \times [0, 1] \to H$ is a near homeomorphism whenever $H$ is an $\ell_2$-model space.*

*Proof.* The identity map $1_H : H \to H$ is a nice map by Lemma 3.12 and $M(1_H)$ is homeomorphic to $H \times [0, 1]$. Hence the projection $\pi_H : H \times [0, 1] \to H$ is a near homeomorphism by Proposition 3.11. □

**Lemma 3.14.** *Let $H$ be an $\ell_2$-model space and $Y$ an $L$-space. If $f : H \to Y$ is a nice cell-like map then $f$ is a near homeomorphism.*

*Proof.* The collapsing map $c_f : M(f) \to Y$, the projection $p_H : H \times [0, 1] \to H$ and the quotient map $\pi : H \times [0, 1] \to M(f)$ are near homeomorphisms by Propositions 3.11, Corollary 3.13 and Propotition 3.6 respectively. Suppose that an open cover $\mathscr{U}$ of $Y$ is given. Let $\mathscr{V}$ be an open cover of $Y$ such that $\mathrm{st}^2 \mathscr{V} \prec \mathscr{U}$. We can take a homeomorphisms $\alpha : M(f) \to Y$ which is $\mathscr{V}$-close to $c_f$, $\beta : H \times [0, 1] \to M(f)$ which is $c_f^{-1}(\mathscr{V})$-close to $\pi$ and $\gamma : H \times [0, 1] \to H$ which is $f^{-1}(\mathscr{V})$-close to $p_H$. Then it follows that $\alpha \circ \beta$ is $\mathrm{st}^2 \mathscr{V}$-close to $f \circ \gamma$. Indeed, for each $x \in H$, $\alpha \circ \beta(x)$ is $\mathscr{V}$-close to $c_f \circ \beta(x)$ which is $\mathscr{V}$-close $c_f \circ \pi(x)$ and $c_f \circ \pi(x) = f \circ p_H(x)$ is $\mathscr{V}$-close to $f \circ \gamma(x)$. Hence, $\alpha \circ \beta \circ \gamma^{-1}$ is $\mathscr{U}$-close to $f$. Thus $f$ is a near homeomorphism. □

**Theorem 3.15** ([1, 3.4 (ii)])**.** *Let $H$ be an $\ell_2$-model space and $Y$ an $L$-space. If $f : H \to Y$ is a cell-like map then $f$ is a near homeomorphism.*

*Proof.* The product map $f \times 1_{\mathcal{Q}} : H \times \mathcal{Q} \to Y \times \mathcal{Q}$ is a nice cell-like map from an $\ell_2$-model space $H \times \mathcal{Q} \approx H$ by Proposition 3.10. Hence $f \times 1_{\mathcal{Q}}$ is a near homeomorphism by Lemma 3.14. In particular, $Y \times \mathcal{Q} \approx H \times \mathcal{Q} \approx H$. So, the projections $p_H : H \times \mathcal{Q} \to H$ and $p_Y : Y \times \mathcal{Q} \to Y$ are nice cell-like maps from $\ell_2$-model space onto $L$-spaces by Lemma 3.12, whence they are near homeomorphisms by Lemma 3.14. Thus $f$ is a near homeomorphism. □

## 4 Cell-like Resolutions

In the following proposition, we assume $X$ to be an $ANR$ though it is not required in the compact setting [1, 4.1]. If we consider the cone over a non-compact space $X$, we always treat *the metrizable cone* $C(X)$, that is, $C(X) = \{*\} \cup (X \times [0, 1))$ equipped with the topology that the open sets of $C(X)$ is generated by the open sets in $X \times [0, 1)$ and the sets $\{*\} \times (X \times (1 - \varepsilon, 1)$, $0 < \varepsilon < 1$.

**Proposition 4.1.** *Let $H$ be an $\ell_2$-model space and $X$ a $Z$-set in $H$. Suppose that $X$ is an $ANR$. Then any proper map $f : X \to H$ can be extended to a cell-like map $\widetilde{f} : H \to H$.*

*Proof.* First we consider the case that $f(X)$ is a $Z$-set in $H$. In this case, using $Z$-set unknotting of $H$, we may assume that $X \cap f(X) = \emptyset$ without loss of generality.

**Claim 1.** There is a $Z$-set $A$ in $H$ which is an $AR$ containing $f(X)$ with $A \cap X = \emptyset$.

Indeed, take a small closed neighborhood $F$ of $f(X)$ in $H$ such that $F \cap X = \emptyset$. Then a $Z$-embeded image of the metrizable cone $C(F)$ is a required $Z$-set $A$ (also we use the $Z$-set unknotting of $H$ to adjust the location of $X$). ♢

Using the strong universality of $H$, we can embed $M(f)$ as a $Z$-set in $H$ so that $A$ and $X$ are identified with the subsets of $H$. Let $c_f : M(f) \to A$ be the collapsing map. Put $Y = H \cup_{c_f} A$ and let $\pi : H \to Y$ be the quotient map.

**Claim 2.** The quotient map $\pi : H \to Y$ is a cell-like map between $AR$s.

It is easy to see that $Y = H \cup_{c_f} A$ is an $ANR$ [5, 6.5.3]. and $\pi$ is a cell-like since the non-trivial part is equal to $c_f : M(f) \to A$. Also, $Y$ is an $AR$ since $\pi$ is a fine homotopy equivalence (Theorem 3.4). ♢

By Proposition 3.5, $\pi : H \to Y$ is a near homeomorphism and $A$ is a $Z$-set in $Y$. Let $i_A : A \to A$ be the identity map from $A \subset Y$ to $A \subset H$. Then we can take a homeomorphism $g : Y \to H$ such that $g \restriction A = i_A$. Indeed, let $h : H \to Y$ be a homeomorphism. Since $Y$ is an $\ell_2$-model space, there is a homeomorphism $l : Y \to H$

such that $l \restriction h(A) = h^{-1} \restriction h(A)$. Then $g = l \circ h$ is a required one. Finally, put $\widetilde{f} = g \circ \pi : H \to H$. Then $\widetilde{f} \restriction X = g \circ f \restriction X = (g \restriction A) \circ f = f$ and clearly $\widetilde{f}$ is a cell-like map. The case $f(X)$ is a $Z$-set in $H$ is proved.

Now we consider the general case. Let $f_0 : X \to H \times [0,1]$ be the map defined by $f_0(x) = (f(x), 0) \in H \times [0,1]$. Then $f_0(X) \subset H \times \{0\}$ is a $Z$-set in $H \times [0,1] \approx H$ (we use the stability of $H$ here). Hence there is a cell-like extension $\widetilde{f_0} : H \to H \times [0,1]$ of $f_0$. Let $p_H : H \times [0,1] \to H$ be the projection. Obviously, $p_H$ is a cell-like map. Then the composition $p_H \circ \widetilde{f_0} : H \to H$ is a required cell-like extension of $f$ since finite composition of cell-like maps between $ANR$s is also a cell-like map [5, 7.5.5]. The proof is finished. □

For a map $f : X \to Y$, let $f_* : X \to Y \times [1,2]$ be the map defined by $f_*(x) = (f(x), 1)$. Then the mapping cylinder $M(f_*)$ is regarded as the union of $M(f)$ and $Y \times [1,2]$ by identifying $Y \subset M(f)$ with $Y \times \{1\} \subset Y \times [1,2]$. We describe the mapping cylinder $M(f_*)$ by $E(f)$ and call $E(f)$ *the extended mapping cylinder of* $f$. Let $c_Y : E(f) \to Y$ be the map defined by $c_Y \restriction M(f) = c_f$ and $c_Y \restriction Y \times [1,2] = p_Y$, where $p_Y : Y \times [1,2] \to Y$ is the projection. By $p_I : E(f) \to [0,2]$ be the projection which maps $M(f)$ onto $[0,1]$ and $Y \times [1,2]$ onto $[1,2]$. For $a < c < b$, we sometime rescale the intervals $[0,1]$ to $[a,c]$ and $[1,2]$ to $[c,b]$, that is, $p_I(M(f)) = [a,c]$ and $p_I(E(f)) = [a,b]$. In this case, $E(f)$ is called *the extended mapping cylinder over* $[a,b]$ *relative to* $[a,c]$.

**Proposition 4.2** ([1, 4.2])**.** *Let $H$ be an $\ell_2$-model space. If $r : H \to A$ is a proper retraction then there is a cell-like map from $H$ onto the extended mapping cylinder $E(r)$ of $r$.*

*Proof.* We note that $r \times 1 : H \times \{1\} \to A \times \{1\}$ is a proper retraction between $ANR$s. Since $H \times \{1\}$ is a $Z$-set in $H$, there is a cell-like extension $f : H \times [1,2] \to H \times [1,2]$ of $r \times 1$ by Proposition 4.1. We identify $M(r) \setminus A \subset E(r)$ with $H \times [0,1)$. Then we define $g : H \times [0,2] \to E(r)$ by $g \restriction H \times [0,1) = \mathrm{id}$ and $g \restriction H \times [1,2] = f$. Using the stability of $H$, we obtain a cell-like map $\widetilde{g} : H \approx H \times [0,2] \to E(r)$. □

**Proposition 4.3.** *Let $A$ be a closed subset of a complete space $X$. If $A$ is an $L$-space, then there is a proper retraction $r : X \to A$.*

*Proof.* Since $A \times [0,1]$ is an $L$-space and $A \times \{0\}$ is a $Z$-set in $A \times [0,1]$, there is a $Z$-embedding $f : X \to A \times [0,1]$ such that $f \restriction A$ is the identity map $A \to A \times \{0\}$. Then $p_A \circ f : X \to A$ is a proper retraction where $p_A : A \times [0,1] \to A$ is the retraction. □

A retraction $r : H \to A$ is called *a convenient retraction* if the fat mapping cylinder $M(r_*)$ of $r$ is homeomorphic to $H$.

**Proposition 4.4.** *Let $H$ be an $\ell_2$-model space and let $r : H \to A$ be a proper retraction. Then the map $r \times 1_{\mathcal{Q}} : H \times \mathcal{Q} \to A \times \mathcal{Q}$ is a proper convenient retraction.*

*Proof.* Since $r \times 1_{\mathcal{Q}} : H \times \mathcal{Q} \to A \times \mathcal{Q}$ is a proper retraction and $H \times \mathcal{Q} \approx H$, there is a cell-like map $f : H \to E(r \times 1_{\mathcal{Q}})$ by Proposition 4.2. If $E(r \times 1_{\mathcal{Q}})$ is an $L$-space, then $f$ is a near homeomorphism and the statement follows by Theorem 3.15. It is easy to see that $E(r \times 1_{\mathcal{Q}})$ is a contractible $ANR$, therefore, an $AR$. So, all we have to see is that $E(r \times 1_{\mathcal{Q}})$ is strongly universal. Let $\mathscr{U}$ and $\mathscr{V}$ be open covers of $E(r \times 1_{\mathcal{Q}})$ with $\mathrm{st}\,\mathscr{V} \prec \mathscr{U}$. Suppose that a map $g : X \to E(r \times 1_{\mathcal{Q}})$ from a complete space $X$ is given. Since $f$ is a fine homotopy equivalence (Theorem 3.4) and $H$ is strongly universal, we can take a closed embedding $h : X \to H$ such that $f \circ h$ is $\mathscr{V}$-close to $g$. Using the fact that $E(r \times 1_{\mathcal{Q}}) \approx E(r) \times \mathcal{Q}$, there is a closed embedding $f' : H \to E(r \times 1_{\mathcal{Q}})$ such that $f'$ is $\mathscr{V}$-close to $f$ by Corollary 3.9. Then the map $g' = f' \circ h : X \to E(r \times 1_{\mathcal{Q}})$ is a closed embedding of $X$ which is $\mathscr{U}$-close to $g$. Hence $E(r \times 1_{\mathcal{Q}})$ is an strongly universal $AR$, that is, an $L$-space. □

Let $r : H \to A$ be a proper retraction. The mapping cylinder $M(r)$ over the interval $[a,b]$ is denoted by $M_r[a,b]$. We use the notation $M_r[a] = H$ and $M_r[b] = A$ in $M_r[a,b]$. If necessary, the sets $A \times \{a\}$, $A \times \{b\} \subset M_r[a,b]$ are denoted by $A[a]$, $A[b]$ respectively. By *the telescope* $M(r,n)$ *of* $n$ *mapping cylinders of* $r$, we mean the adjunction space of $\cup_{i=1}^n M_r[t_{i-1}, t_i]$ obtained by the identification of $A = M_r[t_i] \subset M_r[t_{i-1}, t_i]$ with the subspace $A[t_i] \subset M_r[t_i] \subset M_r[t_i, t_{i+1}]$ for every $i \le n \in \mathbb{N}$, where $t_i = i/n$. The induced collapsing map and projection of $M(r,n)$ are denoted by $c_r^n : M(r,n) \to A$ and $p_I^n : M(r,n) \to [0,1]$ respectively.

Similarly, we define *the infinite telescope* $M(r,\infty)$ *of (mapping cylinders of)* $r$ as the adjunction space of $\cup_{i=1}^\infty M_r[s_{i-1}, s_i]$ using the sequence $0 = s_0 < s_1 < \cdots < s_i \to \infty$. Then the induced collapsing map and projection are denoted by $c_r^\infty : M(r,\infty) \to A$ and $p_I^\infty : M(r,\infty) \to [0,\infty)$ respectively. We some time describe $M(r,n) = \cup_{i=1}^{n} M_r[t_{i-1}, t_i]$ and $M(r,\infty) = \cup_{i=1}^{\infty} M_r[s_{i-1}, s_i]$ to indicate the ingredients of the telescopes.

**Proposition 4.5.** *Let $H$ be an $\ell_2$-model space. If $r : H \to A$ is a proper convenient retraction then $M(r,\infty)$ is homeomorphic to the product space $H \times [0,\infty)$.*

*Proof.* Suppose that $M(r,\infty)$ is given by $\cup_{i=1}^{\infty} M_r[i-1,i]$. Then $(p_I^\infty)^{-1}([0,1/2])$ is homeomorphic to $H \times [0,1] \approx H$ and $(p_I^\infty)^{-1}([i+\frac{1}{2}, i+\frac{3}{2}])$ is homeomorphic to $E(r)$ which is also homeomorphic to $H$ since $r$ is a convenient retraction. Applying $Z$-set unknotting, we obtain a homeomorphism $h : M(r,\infty) \to H \times [0,\infty)$ which maps $(p_I^\infty)^{-1}([0,1/2])$ to $H \times [0,1/2]$ and maps $(p_I^\infty)^{-1}([i+\frac{1}{2}, i+\frac{3}{2}])$ to $H \times [i+\frac{1}{2}, i+\frac{3}{2}]$ for every $i$. □

**Proposition 4.6** ([1, 4.3]). *Let $H$ be an $\ell_2$-model space and $r : H \to A$ a proper convenient retraction. Then there is a homeomorphism $h : M(r) \to M(r,2)$ such that the restrictions $h \restriction A[0] : A[0] \to A[0] \subset M_r[0,\frac{1}{2}]$ and $h \restriction A[1] : A[1] \to A[1] \subset M_r[\frac{1}{2},1]$ are the identity, where $M(r) = M_r[0,1]$ and $M(r,2) = M_r[0,\frac{1}{2}] \cup M_r[\frac{1}{2},1]$. Moreover, $h$ can be choosen so that $c_r^2 \circ h$ is $\mathscr{U}$-close to $c_r$ for each open cover $\mathscr{U}$ of $A$.*

*Proof.* Let $\mathscr{V}$ and $\mathscr{W}$ be coverings of $H$ such that $\mathrm{st}^4 \mathscr{W} \prec \mathscr{V} \prec r^{-1}(\mathscr{U})$. We use the notation as in section 2, that is, $M(r) = H[0,1]$. Then we identify $p_I^{-1}([0,\frac{2}{3}])$ with $H[0,\frac{2}{3}]$ and $(p_I^2)^{-1}([0,\frac{2}{3}])$ with $E(r)$ which is homeomorphic to $H$ since $r$ is a convenient retraction. With this identification, $E(r)$ is the extended mapping cylinder of $r$ over $[0,\frac{2}{3}]$ relative $[\frac{1}{2},\frac{2}{3}]$ and it is denoted by $E[0,\frac{2}{3}]$. Especially, the subspaces $H \times \{0\}$ and $H \times \{\frac{2}{3}\}$ of $E[0,\frac{2}{3}]$ are denoted by $E[0]$ and $E[\frac{2}{3}]$ respectively. Let $c_H : E[0,\frac{2}{3}] \to H$ be the collapsing map and let $p_H : H[0,\frac{2}{3}] \to H$ be the projection. Then $c_H$ and $p_H$ are cell-like maps between $\ell_2$-model spaces, whence they are near homeomorphisms by Theorem 3.15. Let $\alpha : E[0,\frac{2}{3}] \to H$ and $\beta : H[0,\frac{2}{3}] \to H$ be homeomorphisms such that $\alpha$ and $\beta$ are $\mathscr{W}$-close to $c_H$ and $p_H$ respectively. We define a homeomorphism $\gamma : H[0,\frac{2}{3}] \to E[0,\frac{2}{3}]$ by $\gamma = \alpha^{-1} \circ \beta$. Then $c_H \circ \gamma : H[0,\frac{2}{3}] \to H$ is $\mathrm{st}\,\mathscr{W}$-close to $p_H$. Let $i : E[0] \to H[0] \subset H[0,\frac{2}{3}]$ and $j : E[\frac{2}{3}] \to H[\frac{2}{3}] \subset H[0,\frac{2}{3}]$ be the identity maps. Then $\gamma \circ i$ and $\gamma \circ j$ are $c_H^{-1}(\mathrm{st}\,\mathscr{W})$-close to the identity maps. Hence, there is a homeomorphism $\xi : E[0,\frac{2}{3}] \to E[0,\frac{2}{3}]$ which is $c_H^{-1}(\mathrm{st}\,\mathscr{W})$-close to the identity such that $\xi \restriction \gamma(H[0]) = (\gamma \circ i)^{-1}$ and $\xi \restriction \gamma(H[\frac{2}{3}]) = (\gamma \circ j)^{-1}$. Then $\eta = \xi \circ \gamma : H[0,\frac{2}{3}] \to E[0,\frac{2}{3}]$ is a homeomorphism such that $c_H \circ \eta$ is $\mathscr{V}$-close to $p_H$. Moreover, $\eta \restriction H[0] = i^{-1} : H[0] \to E[0]$ and $\eta \restriction H[\frac{2}{3}] = j^{-1} : H[\frac{2}{3}] \to E[\frac{2}{3}]$. Now we define $h : M(r) \to M(r,2)$ by $h \restriction H[0,\frac{2}{3}] = \eta$ and $h \restriction H[\frac{2}{3},1]$ is the natural identification of $H[\frac{2}{3},1]$ and $(p_I^2)^{-1}([\frac{2}{3},1])$. Then $h$ is a well-defined homeomorphism such that $c_r^2 \circ h$ is $\mathscr{U}$-close to $c_r$ since $\mathscr{V} \prec r^{-1}(\mathscr{U})$. Now it is obvious that $h \restriction A[0]$ and $h \restriction A[1]$ are the identity map. □

**Lemma 4.7.** *Let $f : X \to Y$ be a proper map. For a given $\varepsilon > 0$, there is an open covering $\mathscr{U}$ of $Y$ such that the fibers $f^{-1}(y_1)$ and $f^{-1}(y_2)$ of $\mathscr{U}$-close two points $y_1$, $y_2 \in Y$ are contained in their $\varepsilon$-neighborhoods one another.*

*Proof.* Consider the hyperspaces $2^X$ and $2^Y$ with the Hausdorff metric. Since $f$ is a proper map, $\mathcal{A} = \{f^{-1}(y) \mid y \in Y\}$ is a subspace of $2^X$. The hyperspace map $2^f$ maps $\mathcal{A}$ onto the subspace $\mathcal{Y} = \{\{y\} \mid y \in Y\} \subset 2^Y$. Let $F = 2^X \restriction \mathcal{A} : \mathcal{A} \to \mathcal{Y}$. Then $F$ is in fact a homeomorphism. Hence we can take an open cover $\mathscr{U}$ of $\mathcal{Y}$ so that $F^{-1}(\mathscr{U}) \prec \{B_{d_H}(A,\varepsilon) \mid A \in \mathcal{A}\}$. □

Let $f : X \to Y$ be a proper map and let $\mathscr{U}$ be an open covering of $Y$. If $\mathscr{U}$ satisfies the condition stated in Lemma 4.7, then we say $\mathscr{U}$ satisfies the property $\mathcal{P}(f,\varepsilon)$.

**Proposition 4.8** ([1, 4.4]). *Let $H$ be an $\ell_2$-model space. If there is a proper convenitent retraction $r : H \to A$ then it induces a cell-like map $\Psi : \mathrm{cone}(H) \to \mathrm{cone}(A)$ (between the metrizable cones).*

*Proof.* It suffices to construct a cell-like map $\Phi : M(r,\infty) \to A \times [0,\infty)$ since $M(r,\infty)$ is homeomorphic to $H \times [0,\infty)$ by Proposition 4.5. Then we obtain a cell-like map $\Phi' : H \times [0,\infty) \to A \times [0,\infty)$, whence a cell-like map $\Psi : \mathrm{cone}(H) \to \mathrm{cone}(A)$ is defined as its one point compactification.

Let $M_n(i)$ be the mapping cylinder of $r$ over $[\frac{i-1}{2^n}, \frac{i}{2^n}]$, $i = 1,2,\cdots$, and let $M_n = \cup_{i=1}^{\infty} M_n(i)$ be the infinite telescope of $r$. Then $M_0 = M(r,\infty)$ and $M_0$ contains the telescope $M(r,n)$ of $n$ mapping cylinders of $r$. Let $c_n : M_n \to A$ be the collapsing map and $p_n : M_n \to [0,\infty)$ the projection. We note that $A$ is an $AR$ as a retract of an $AR$ space $H$, whence $A$ is locally contractible. So, we can take an open covering $\mathscr{E}_1$ of $A$ such that $\mathrm{mesh}\,\mathscr{E}_1 < 2^{-1}$ and every element of $\mathscr{E}_1$ can be contracted to a point in some set with diameter $< 2^{-1}$. Since $c_0 \restriction M(r,1) : M(r,1) \to A$ is a proper map, taking a refinement if necessary, we may assume that $\mathscr{E}_1$ satisfies the property $\mathcal{P}(c_0 \restriction M(r,1), 2^{-1})$ by Lemma 4.7. By Proposition 4.6, we can take a homeomorphism $f_0^1 : M_0 \to M_1$ such that $c_1 \circ f_0^1$ is $\mathscr{E}_1$-close to $c_0$ and $f_0^1(M_0(i)) = M_1(2i-1) \cup M_1(2i)$ for every $i \in \mathbb{N}$. Suppose that homeomorphisms $f_{i-1}^i : M_{i-1} \to M_i$ and open covers $\mathscr{E}_i$ of $A$ have been constructed for $i = 1, \cdots, n$. Let $f_0^n = f_{n-1}^n \circ \cdots \circ f_0^1 : M_0 \to M_n$ be the composition. For each $n \in \mathbb{N}$, we take an open covering $\mathscr{E}_{n+1}$ of $A$ satisfying the following:

(1) $\operatorname{st} \mathscr{E}_{n+1} \prec \mathscr{E}_n$,
(2) $\operatorname{mesh} \mathscr{E}_{n+1} < 2^{-(n+1)}$,
(3) $\mathscr{E}_{n+1}$ satisfies the property $\mathcal{P}(c_n \circ f_0^n \restriction M(r,n), 2^{-(n+1)})$ and
(4) every element of $\mathscr{E}_{n+1}$ can be contracted to a point in some set with diameter $< 2^{-(n+1)}$.

The condition (3) is assured by Lemma 4.7 since $c_n \circ f_0^n \restriction M(r,n) : M(r,n) \to A$ is a proper map. Then, using Proposition 4.6, we can take a homeomorphism $f_n^{n+1} : M_n \to M_{n+1}$ such that

(5) $c_{n+1} \circ f_n^{n+1} : M_n \to A$ is $\mathscr{E}_{n+1}$-close to $c_n$ and
(6) $f_n^{n+1}(M_n(i)) = M_{n+1}(2i-1) \cup M_1(2i)$ for every $i \in \mathbb{N}$.

Let $\gamma_n : M_0 \to A$ and $\pi_n : M_0 \to [0,\infty)$ be the collapsing map and the projection through $f_0^n : M_0 \to M_n$ respectively, that is, $\gamma_n = c_n \circ f_0^n$ and $\pi_n = p_n \circ f_0^n$. Then we have

(3)′ $\mathscr{E}_{n+1}$ satisfies the property $\mathcal{P}(\gamma_n \restriction M(r,n), 2^{-(n+1)})$ and
(5)′ $\gamma_{n+1}$ is $\mathscr{E}_{n+1}$-close to $\gamma_n$.

**Claim 1.** The sequences of maps $\{\gamma_n\}$ and $\{\pi_n\}$ are uniformly convergence.

Indeed, suppose that a point $x \in M_0$ and a number $n$ are given. Let $i$ be the number such that $f_0^n(x) \in M_n(i)$. Then $\pi_m(x) \in [\frac{i-1}{2^n}, \frac{i}{2^n}]$ for $\forall m \geq n$ by the condition (6). So we have $d(\pi_n(x), \pi_m(x)) < 2^{-n}$. Also, we have $d(\gamma_n(x), \gamma_{n+1}(x)) = d(c_n \circ f_0^n(x), c_{n+1} \circ f_n^{n+1} \circ f_0^n(x)) < 2^{-(n+1)}$ by condition (5) and (2). Thus, $d(\gamma_n(x), \gamma_m(x)) \leq 2^{-(n+1)}/(1-2^{-1}) = 2^{-n}$. ♢

Let $\gamma : M_0 \to A$ and $\pi : M_0 \to [0,\infty)$ be the uniform limits of $\{\gamma_n\}$ and $\{\pi_n\}$ respectively. Then we define $\Phi : M_0 \to A \times [0,\infty)$ by $\Phi(x) = (\gamma(x), \pi(x))$, $x \in M_0$. Similarly, we define $\Phi_n : M_0 \to A \times [0,\infty)$ by $\Phi_n(x) = (\gamma_n(x), \pi_n(x))$, $x \in M_0$, for each $n$. Clearly, $\{\Phi_n\}$ uniformly convergent to $\Phi$, i.e., $\lim_{n\to\infty} \Phi_n = \Phi$.

**Claim 2.** $\Phi$ is a proper map.

Let $K$ be a compact subset of $A \times [0,1)$. Then there are a compact subset $K_A \subset A$ and an integer $k$ such that $K \subset K_A \times [0,k]$. Note that $\Phi^{-1}(K) \subset \Phi^{-1}(K_A \times [0,k]) \subset \gamma^{-1}(K_A) \cap \pi^{-1}[0,k] \subset M(r,k)$. Similarly, $\Phi_n^{-1}(K) \subset \gamma_n^{-1}(K_A) \cap M(r,k)$. Then $\Phi_n \restriction M(r,k) : M(r,k) \to A \times [0,1)$ is a proper map since $\gamma_n \restriction M(r,k)$ is a proper map. Thus, $\Phi_n^{-1}(K)$ is compact for every $n \geq k$. Suppose $n \geq k$ and denote the restriction $\gamma_n \restriction M(r,k)$ by $\phi_n : M(r,k) \to A$ for notational simplicity. By the conditions (3) and (5), the property $\mathcal{P}(\phi_n, 2^{-(n+1)})$ implies that $d_H(\phi_n^{-1}(a), \phi_{n+1}^{-1}(a)) < 2^{-(n+1)}$ for every $a \in A$. Hence the family $\{\Phi_n^{-1}(K)\}$ is a Cauchy sequence in $2^{M_0}$ with $\lim_{n\to\infty} \Phi_n^{-1}(K) = \Phi^{-1}(K)$ (c.f. [4, 1.11.2]). Thus $\Phi^{-1}(K)$ is compact. ♢

Now we shall check that $\Phi$ is a cell-like map. Let $x = (a,t) \in A \times [0,\infty)$ and $F = \Phi^{-1}(x)$.

**Claim 3.** For each $n \in \mathbb{N}$, there is $i \in \mathbb{N}$ such that $f_0^n(F) \subset M_n(i) \cup M_n(i+1)$.

Indeed, if $f_0^n(F)$ is contained in three mapping cylinders, then the diameter of $p_m \circ f_0^m(F)$ must be greater than $2^{-n}$ for every $m \geq n$ by (6). Then the limit $\pi(F) = \lim_{n\to\infty} \pi_n(F)$ cannot be the one point $t$. ♢

**Claim 4.** For each $n \in \mathbb{N}$, $\gamma_n(x)$ and $\gamma(x) = a$ are $\mathscr{E}_{n-1}$-close for every $x \in F$. In particular, $\gamma_n(F)$ is contained in an element of $\mathscr{E}_{n-2}$.

Suppose $m > n$ and let $x \in F$. By (5)′, $\gamma_m(x)$ and $\gamma_{m-1}(x)$ are $\mathscr{E}_m$-close. Then $\gamma_m(x)$ and $\gamma_{m-2}(x)$ are $\mathscr{E}_{m-2}$-close since $\gamma_{m-1}(x)$ and $\gamma_{m-2}(x)$ are $\mathscr{E}_{m-1}$-close and $\operatorname{st} \mathscr{E}_{m-1} \prec \mathscr{E}_{m-2}$. Suppose that $\gamma_m(x)$ and $\gamma_{n+1}(x)$ are $\mathscr{E}_{n+1}$-close. Then $\gamma_m(x)$ and $\gamma_n(x)$ are $\mathscr{E}_n$-close since $\gamma_{n+1}(x)$ and $\gamma_n(x)$ are $\mathscr{E}_{n+1}$-close and $\operatorname{st} \mathscr{E}_{n+1} \prec \mathscr{E}_n$. Thus $\gamma(x) = a$ and $\gamma_n(x)$ are $\mathscr{E}_{n-1}$-close. Hence $\gamma_n(F)$ is contained in an element of $\operatorname{st} \mathscr{E}_{n-1} \prec \mathscr{E}_{n-2}$. ♢

Let $U$ be an closed neighborhood of $\gamma_n(F)$ such that $\operatorname{diam} U < 2^{-(n-2)}$ and $\gamma_n(F)$ can be contracted to a point in $U$. Suppose that $f_0^n(F) \subset M_n(i) \cup M_n(i+1)$.

**Claim 5.** $f_0^n(F)$ can be contracted to a point in $c_n^{-1}(U) \cap (M_n(i) \cup M_n(i+1))$.

Indeed, we contract $f_0^n(F)$ to $\gamma_n(F) \times \{\frac{i+1}{2^n}\} \subset A[\frac{i+1}{2^n}] \subset M_n(i+1)$ by the collapsing map $M_n(i) \cup M_n(i+1) \to A$. The collapsing is done in the set $c_n^{-1}(U) \cap (M_n(i) \cup M_n(i+1))$. Then it is contracted to a point in $U \times \{\frac{i+1}{2^n}\} \subset A[\frac{i+1}{2^n}] \subset M_n(i+1)$. ♢

Since $\gamma$ and $\gamma_n$ are $\mathscr{E}_{n-1}$-close and the diameter of $U$ is smaller than $2^{-(n-2)}$, the diameter of $\gamma(\gamma_n^{-1}(U))$ is smaller than $2^{-(n-3)}$. Hence $F$ can be contracted to a point in $\Phi^{-1}(B(a, 2^{-(n-3)}) \times [\frac{i-1}{2^n}, \frac{i}{2^n}])$. Since we take $n$ arbitrary and $\Phi$ is a proper map, this means that $F$ can be contracted to a point in any neighborhood of itself, that is, $\Phi$ is a cell-like map. □

## 5 Topological characterization of Hilbert space

In this section, the cone over a space $X$ always means the metrizable cone over $X$. One should note that the natural map $\pi : X \times [0,1] \to \mathrm{cone}(X)$ sending $X \times \{1\}$ to the cone point $\{*\}$ is a quotient map if $X$ is compact. However, it is not a quotient map whenever $X$ is not compact.

**Proposition 5.1.** *Let $H$ be an $\ell_2$-model space and let $\mathrm{cone}(H)$ be the metrizable cone over $H$. If $\pi : H \times [0,1] \to \mathrm{cone}(H)$ is the natural map sending $H \times \{1\}$ to the cone point $\{*\}$ then $\pi$ is a near homeomorphism.*

*Proof.* Let $\mathscr{U}$ and $\mathscr{V}$ be given open covers of $H \times [0,1]$ and $\mathrm{cone}(H)$ respectively. We shall construct a homeomorphism $h : H \times [0,1] \to H \times [0,1]$ such that $\{h(\pi^{-1}(y)) \mid y \in \mathrm{cone}(H)\} \prec \mathscr{U}$ and $\pi \circ h$ is $\mathscr{V}$-close to $\pi$. Then the statement follows from Bing's shrinking criterion 2.7. Take $\varepsilon > 0$ so that the neighborhood $\{*\} \cup (H \times (1-2\varepsilon, 1))$ is contained in some element of $\mathscr{V}$. Let $x \in H \times (1-\varepsilon, 1) \subset H \times [0,1]$ be a point and let $U$ be a neighborhood of $x$ in $H \times [0,1]$ which is contained in some element of $\mathscr{U}$. Then the collapsing map $c : H \times \{1\} \to \{*\}$ can be approximated by a $Z$-embedding $f : H \times \{1\} \to H \times [0,1] \approx H$ which is homotopic to $c$ in $H \times (1-\varepsilon, 1)$. By the $Z$-set unknotting theorem, there is a homeomorphism $h : H \times [0,1] \to H \times [0,1]$ supported by $H \times (1-\varepsilon, 1)$ and $h \restriction H \times \{1\} = f$. □

**Theorem 5.2** (cf. [1, 3.5])**.** *Any $L$-space is a cell-like image of any $\ell_2$-model space.*

*Proof.* Let $A$ be an $L$-space and $H$ an $\ell_2$-model space. We shall show that $A \times \mathcal{Q} \approx H$. Then the projection $p_A : A \times Q \to A$ is a required cell-like map.

Without loss of generality, we may assume that $A$ is a closed subset of $H$. By Proposition 4.3, there is a proper retraction $r : H \to A$. Then $r \times 1_{\mathcal{Q}} : H \times \mathcal{Q} \to A \times \mathcal{Q}$ is a proper convenient retraction by Proposition 4.4. Since $H \times \mathcal{Q} \approx H$, there is a cell-like map $\Psi : \mathrm{cone}(H) \to \mathrm{cone}(A \times \mathcal{Q})$ by Proposition 4.8. Hence $\Psi \times 1_{\mathcal{Q}} : \mathrm{cone}(H) \times \mathcal{Q} \to \mathrm{cone}(A \times \mathcal{Q}) \times \mathcal{Q}$ is also a cell-like map. One should note that $\mathrm{cone}(A \times \mathcal{Q})$ is an $L$-space (cf. [5, 6.5.7]) and $\mathrm{cone}(H) \times \mathcal{Q} \approx H$ by Proposition 5.1. So the map $\Psi \times 1_{\mathcal{Q}}$ is a cell-like map from an $\ell_2$-model space onto an $L$-space, that is, $\Psi \times 1_{\mathcal{Q}} : \mathrm{cone}(H) \times \mathcal{Q} \to \mathrm{cone}(A \times \mathcal{Q}) \times \mathcal{Q}$ is a near homeomorphism by Theorem 3.15. As a result, we have $\mathrm{cone}(A \times \mathcal{Q}) \times \mathcal{Q} \approx \mathrm{cone}(H) \times \mathcal{Q} \approx H \times \mathcal{Q} \approx H$. Considering the natural projection $((A \times \mathcal{Q}) \times [0,1]) \times \mathcal{Q} \to \mathrm{cone}(A \times \mathcal{Q}) \times \mathcal{Q} \approx H$, we can see that $((A \times \mathcal{Q}) \times [0,1]) \times \mathcal{Q}$ is an $H$-manifold (locally homeomorphic to open subsets of $H$). Hence, the space $A \times \mathcal{Q} \approx ((A \times \mathcal{Q}) \times [0,1]) \times \mathcal{Q}$ is an $\ell_2$-model space. In fact, the universality follows from the $A$'s universality and $Z$-set unknotting follows from the open embedding theorem (cf. [6, 2.5.10]). So we have $\mathrm{cone}(A \times \mathcal{Q})$ is homeomorphic to $A \times \mathcal{Q}$ by Proposition 5.1. Therefore, $A \times \mathcal{Q} \approx A \times \mathcal{Q} \times \mathcal{Q} \approx \mathrm{cone}(A \times \mathcal{Q}) \times \mathcal{Q} \approx H$. The proof is finished. □

**Theorem 5.3** (Characterization of Hilbert space)**.** *Any $L$-space is homeomorphic to any $\ell_2$-model space. In particular, any complete strongly universal AR space is homeomorphic to the Hilbert space $\ell_2$.*

*Proof.* The first part follows from Theorem 3.15 and Theorem 5.2. The second part follow from the fact that $\ell_2$ is an $\ell_2$-model space. □

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